3) In the above circuit, use $V_{DD}=1.8 \text{ V}$, $I_{SS}=14 \mu \text{A}$, $R_D=100 \text{ k}\Omega$, $W=10 \mu \text{m}$, $L=1 \mu \text{m}$.
a) Calculate $V_{DS}$ of the two transistors by hand, with $V_{ic}=0.9 \text{ V}$, $V_{id}=0 \text{ V}$, and verify with SPICE.

$$V_{o1} = V_{o2} = V_{DD} - \frac{I_{SS}}{2} R_D = 1.1 \text{ V}$$

$$V_{DSAT} = \sqrt{\frac{2I_{DS}}{k'\frac{W}{L}}} = \sqrt{\frac{I_{SS}}{k'\frac{W}{L}}} = 0.1 \text{ V}$$

$$V_S = V_{ic} - V_T - V_{DSAT}$$

$$= V_{ic} - (V_{T0} + \gamma(\sqrt{2\phi + V_S} - \sqrt{2\phi})) - V_{DSAT}$$

$$= 0.9 - (0.5 - 0.5(\sqrt{0.6 + V_S} - \sqrt{0.6})) - 0.1$$

$$= 0.3 - 0.5(\sqrt{0.6 + V_S} - \sqrt{0.6})$$

Solving iteratively, we find $V_S = 0.231 \text{ V}$, and therefore, $V_{DS} = 0.87 \text{ V}$.
b) Plot $V_{od} = (V_{o1} - V_{o2})$ vs. $V_{id}$, with $V_{ic} = 0.9$ V over the range $-1.8 < V_{id} < +1.8$ V.

The slope around $V_{id} = 0$ V is $A_{m} = -g_{m} R_{out} \approx -g_{m} R_{D} = \left(-\sqrt{2k' \frac{W}{L} I_{DS}}\right) R_{D} = -14$.

The curves limit when all of the current is switched to one side of the differential pair. When this happens, the differential voltage is $I_{SS} R_{D} = 1.4$ V.

The voltage that this occurs at can be approximated by $\Delta V_{id} = \frac{I_{SS} R_{D}}{g_{m} R_{D}} = 0.1$ V.

It is also correct to find the exact point where one transistor turns off and the other takes all of the current. This occurs for $\Delta V_{id} = \sqrt{\frac{2I_{SS}}{k' \frac{W}{L}}} = 0.14$ V.
c) Plot $V_{oc} = \frac{V_{o1} + V_{o2}}{2}$ vs. $V_{ic}$, with $V_{id}=0$ V over the range $0 < V_{ic} < +1.8$ V.

There are no breakpoints in this curve. The strange behavior near $V_{oc}=0$ V is only an artifact of the ideal current source pulling the source voltage below ground. When $V_S<0$ V, the source-bulk junction diode begins to be forward biased and starts to conduct current. This will not happen with any real implementation, and you do not have to worry about calculating this.

d) Calculate $A_{dm}$ with $V_{ic}=0.9$ V, $V_{id}=0$ V. Over what range of $V_{id}$ will the gain remain high? Why does the gain drop off?

$$A_{dm} = -g_m (R_D \| r_v) = -g_m R_D$$

$$= -\sqrt{2(140 \mu A/V^2)(10)(70 \mu A)(100 k)}$$

$$= -14$$

As calculated in part (b), the gain is high over the range of $-0.1 < V_{id} < 0.1$ V. The gain drops off because one of the transistors goes into cutoff.
e) Calculate $A_{cm}$ with $V_{ic}=0.9$ V, $V_{id}=0$ V.

$$A_{cm} \approx -\frac{g_m R_D}{1 + (1 + \chi) g_m (2 r_o)}$$

For an ideal current source, the output resistance is $r_o = \infty$, and therefore the common-mode gain is, $A_{cm} = 0$.

f) Calculate $R_{od}$ with $V_{ic}=0.9$ V, $V_{id}=0$ V.

$$R_{od} = 2(R_{D} || r_o) = 2(100 \text{ k} || 1.4 \text{ M})$$

$$= 187 \text{ k}\Omega$$

Verify (d)-(f) with SPICE using the .TF analysis option.
* hw3, 3: diff pair w/ ideal tail current

.model nch nmos level=1 tox=25 vto=0.5 kp=140e-6 lambda=0.1 gamma=0.5 phi=0.6

vdd vdd 0 1.8

* set up common mode and differential inputs

vic vic 0 0.9
vid vid 0 0
e1 v11 vic vid 0 0.5
e2 v12 vic vid 0 -0.5

* set up to measure output common mode

e3 voc1 0 vol 0 0.5
e4 voc voc1 vo2 0 0.5
rcm voc 0 1

rd1 vdd vol 100k
rd2 vdd vo2 100k
m1 vol v11 s 0 nch w=10u l=1u
m2 vo2 v12 s 0 nch w=10u l=1u
iss s 0 14u

.options post=2 nomod

.probe dc vod=v(vol,vo2) voc=v(voc)

.op
.dc vid -1.8 1.8 0.01
dc vic 0 1.8 0.01
tf v(vol,vo2) vid
.alter
tf v(voc) vic
.end

**** mosfets

subckt
element 0:m1 0:m2
model 0:nch 0:nch
id 7.0000u 7.0000u
vgs 665.4148m 665.4148m
vds 865.4147m 865.4147m Vds
vth 569.4798m 569.4798m
vdsat 95.9349m 95.9349m

**** small-signal transfer characteristics

\[
\frac{v(vol,vo2)/vid}{\text{Adm}} = -13.7100
\]
\[
\text{input resistance at vid} = 1.000e+20
\]
\[
\text{output resistance at v(vol,vo2)} = 187.8949k \text{ Rod}
\]

**** small-signal transfer characteristics

\[
\frac{v(voc)/vic}{\text{Acm}} = -79.0254n
\]
\[
\text{input resistance at vic} = 1.000e+20
\]
\[
\text{output resistance at v(voc)} = 0.
\]
The schematic for the circuit used in Problem is:
replace the $I_{SS}$ current source with a resistor which results in the same $I_{DS}$ currents when $V_{ic}=0.9$ V and $V_{id}=0$ V.

a) To get the same current, 

$$R_{SS} = \frac{V_S}{I_{SS}} = \frac{0.231 \text{ V}}{14 \mu\text{A}} = 16.5 \text{k}\Omega$$

Since the current and source voltage are the same, $V_{DS}$ is also the same.

$$V_{DS} = 0.87 \text{ V}$$

b) Plot $V_{od}$ vs. $V_{id}$ with $V_{ic}=0.9$ V over the range $-1.8 < V_{id} < 1.8$ V.

Differentially, this circuit behaves the same way as the circuit with an ideal current source. Therefore the breakpoints are the same as before:

$$\Delta V_{id} = \pm 0.1 \text{ V}, \ \Delta V_{od} = \pm 1.4 \text{ V}$$

When $V_{id}$ is increased beyond this range, the current increases causing a small slope in the transfer function. This can be approximated by a flat line and the exact shape is not important.
c) Plot $V_{oc}$ vs. $V_{ic}$ with $V_{id}$=0 V over the range $0 < V_{ic} < +1.8$ V.

The equivalent half-circuit is

When $V_{ic} < V_T$, the transistor is operating in cutoff and the output is at $V_{DD}$=1.8 V.

When $V_{ic} > V_T$, the transistor operates as a common-source with source-degeneration amplifier until it goes into triode. We need to find the point where this occurs. At the edge of saturation,
Solving the quadratic equation, \( I_{DS} = 12.5 \, \mu A \), and \( V_{DSAT} = 0.134 \, V \).

Therefore, \( V_S = I_{DS}(2R_{SS}) = 0.413 \, V \), and \( V_T = 0.616 \, V \). So, the transistor goes from saturation to triode at the point,

\[
V_{ic} = V_S + V_T + V_{DSAT} = 1.16 \, V
\]
\[
V_{oc} = V_S + V_{DSAT} = 0.547 \, V
\]

d) Since the differential half-circuit is exactly the same as with the ideal tail current source, this is the same as in (3d).

\[
A_{dM} = -g_m(R_D || r_o) = -14
\]

and the gain is large over the range \(-0.1 \, V < V_{id} < 0.1 \, V\). The gain drops off because one of the transistors goes into cutoff.

e) The common-mode gain is slightly different due to the source resistor.

\[
A_{cm} = -\frac{g_m R_D}{1 + (1 + \gamma)g_m(2R_{SS})} = -2
\]

where

\[
g_m = 140 \, \mu S
\]
\[
\gamma = \frac{\gamma}{2(2\phi + V_{SB})^{1/2}} = 0.274
\]

f) The differential output resistance is the same as in (4f),

\[
R_{od} = 2(R_D || r_o) = 187 \, k\Omega
\]
** hw3, 4: diff pair w/ tail resistor

.model nch nmos level=1 tox=25 vto=0.5 kp=140e-6 lambda=0.1 gamma=0.5 phi=0.6

vdd vdd 0 1.8

* set up common mode and differential inputs
vic vic 0 0.9
vid vid 0 0
e1 vi1 vic vid 0 0.5
e2 vi2 vic vid 0 -0.5

* set up to measure output common mode
e3 voc1 0 vol 0 0.5
e4 voc voc1 vo2 0 0.5
rcm voc 0 1

rd1 vdd vol 100k
rd2 vdd vo2 100k
ml vol vi1 s 0 nch w=10u l=1u
m2 vo2 vi2 s 0 nch w=10u l=1u
rss s 0 16.5k

.options post=2 nomod

.probe dc vod=v(vol,vo2) voc=v(voc)

.op
.dc vid -1.8 1.8 0.01
.dc vic 0 1.8 0.01
.tf v(vol,vo2) vid
.alter
.tf v(voc) vic

.end

**** mosfets

subckt
element 0:m1 0:m2
model 0:nch 0:nch
id 7.0927u 7.0927u
vgs 665.9422m 665.9422m
\textbf{vds} 856.6759m 856.6759m 856.6759m V\textsubscript{ds}
vth 569.3355m 569.3355m
vdsat 96.6067m 96.6067m

**** small-signal transfer characteristics

\begin{align*}
\text{v(vol,vo2)/vid} &= -13.7831 \quad \text{Adm} \\
\text{input resistance at vid} &= 1.000e+20 \\
\text{output resistance at v(vol,vo2)} &= 187.7352k \quad \text{Rod}
\end{align*}

**** small-signal transfer characteristics

\begin{align*}
\text{v(voc)/vic} &= -2.0228 \quad \text{Acm} \\
\text{input resistance at vic} &= 1.000e+20 \\
\text{output resistance at v(voc)} &= 0.
\end{align*}
The schematic for the circuit used is:
(a) \[ V_X = \frac{V_{out} - V_{in}}{r_{o1}} - G_{m1} V_{in} = -\frac{V_{out}}{r_{o3} + r_{o2} (1 + G_{m2} \cdot r_{o3})} \]

\[ V_{out} \left[ \frac{1}{r_{o1}} + \frac{1}{r_{o3} + r_{o2} (1 + G_{m2} \cdot r_{o3})} \right] = (G_{m1} + \frac{1}{r_{o1}}) V_{in} \]

\[ V_{out} = \frac{V_{in}}{r_{o1} + r_{o3} + r_{o2} (1 + G_{m2} \cdot r_{o3})} \]

(b) \[ G_m = \frac{G_{m2} \cdot r_{o2}}{\left(\frac{1}{G_{m1}} - \frac{1}{r_{o1}}\right) + \left(1 + G_{m2} \left(\frac{1}{G_{m1}} \cdot r_{o1}\right)\right) \cdot r_{o2}} \]

\[ R_{out} = \left(\frac{1}{G_{m3}} \cdot r_{o3}\right) \cdot \left\{ \left[1 + G_{m2} \left(\frac{1}{G_{m1}} \cdot r_{o1}\right)\right] \cdot r_{o2} + \left(\frac{1}{G_{m1}} \cdot r_{o1}\right) \right\} \]

\[ A_V = G_m \cdot R_{out} = \frac{G_{m2} \cdot r_{o2} \left(\frac{1}{G_{m3}} \cdot r_{o3}\right)}{\left(\frac{1}{G_{m3}} \cdot r_{o3}\right) + \left[1 + G_{m2} \left(\frac{1}{G_{m1}} \cdot r_{o1}\right)\right] \cdot r_{o2} + \left(\frac{1}{G_{m1}} \cdot r_{o1}\right)} \]

Resistance seen looking up at the source of \( M_2 \), \( \frac{1}{G_{m3} \cdot r_{o3}} + r_{o2} \)

\[ \frac{V_{out}}{V_{in}} = \frac{\left(\frac{1}{G_{m1}} \cdot r_{o1}\right)}{\left(\frac{1}{G_{m1}} \cdot r_{o1}\right) + \left(\frac{1}{G_{m3}} \cdot r_{o3}\right) + r_{o2}} \]

\[ \frac{1 + G_{m2} \cdot r_{o3}}{1 + G_{m2} \cdot r_{o3}} \]
\[ G_m = \frac{G_m \cdot r_{o2}}{r_{o1} + \left(1 + G_m \cdot r_{o1}\right) r_{o2}} \]
\[ R_{out} = \frac{r_{o3}}{1 + \left(1 + G_m \cdot r_{o1}\right) r_{o2} + r_{o1}} \]
\[ \frac{V_{out}}{V_{in}} = \frac{G_m \cdot r_{o2} \cdot r_{o3}}{r_{o3} + (1 + G_m \cdot r_{o1}) r_{o2} + r_{o1}} \]

Resistance seen looking up at the source of \( M_2 \)
\[ R_{in} = \frac{r_{o3} + r_{o2}}{1 + G_m \cdot r_{o2}} \]
\[ \frac{V_{out}}{V_{in}} = \frac{\frac{r_{o1}}{r_{o1} + \frac{r_{o3} + r_{o2}}{1 + G_m \cdot r_{o2}}}}{\frac{r_{o1} (1 + G_m \cdot r_{o2})}{r_{o1} (1 + G_m \cdot r_{o2}) + r_{o2} + r_{o3}}} \]

\[ \begin{align*}
1) & \quad -\left(\frac{V_{out}}{r_{o3}} + G_m V_X\right) = \left(\frac{V_{out} - V_X}{r_{o2}} - G_m V_X\right) = \frac{V_X}{r_{o1}} + G_m V_{in} \\
2) & \quad \frac{V_X}{r_{o3}} + G_m V_X = \frac{V_{out} - V_X}{r_{o2}} = V_X = \frac{1}{r_{o2}} + \frac{1}{r_{o3}} - \frac{V_{out}}{r_{o2} + G_m - G_m} \\
3) & \quad \frac{V_{out}}{r_{o3}} - G_m V_X = \frac{V_X}{r_{o1}} + G_m V_{in} \\
4) & \quad \left[\frac{1}{r_{o3}} + \frac{G_m + r_{o1}}{r_{o2}}\right] = \frac{1}{r_{o2} + G_m - G_m} - \frac{V_{out}}{r_{o3}} = G_m \cdot V_{in} \\
5) & \quad -\frac{V_{out}}{r_{o3}} + \frac{G_m + r_{o1}}{r_{o2}} \left[\frac{1}{r_{o2} + G_m - G_m} \right] = G_m \cdot V_{in} \\
6) & \quad -\frac{V_{out}}{r_{o3}} + \frac{G_m + r_{o1}}{r_{o2}} \left[\frac{1}{r_{o2} + G_m - G_m} \right] = G_m \cdot V_{in} \\
\end{align*} \]
\[-V_{out} \left[ \frac{1}{r_{o1}} + \frac{(1 + \alpha_m r_{o1})(r_{o2} + r_{o3})}{r_{o1} r_{o3} \left( 1 + (\alpha_m - \alpha_m_3) r_{o2} \right)} \right] = \alpha_m \cdot V_{in} \]

\[\frac{-V_{out}}{V_{in}} = \frac{1}{r_{o1} \left[ 1 + (\alpha_m - \alpha_m_3) r_{o2} \right] + (r_{o2} + r_{o3}) \left( 1 + \alpha_m_3 r_{o1} \right)}\]

\[V_{x} = \frac{1}{r_{o2}} + \frac{\alpha_m - \alpha_m_3}{1 + \frac{1}{r_{o3}}} \cdot V_{out} \]

\[-\frac{V_{x}}{r_{o3}} + \alpha_m_3 V_{out} = \frac{V_{out}}{r_{o1}} + \alpha_m \cdot V_{in}\]

\[V_{out} = \frac{\alpha_m r_{o1} (r_{o2} + r_{o3})}{r_{o1} \left[ 1 + (\alpha_m - \alpha_m_3) r_{o2} \right] + (r_{o2} + r_{o3}) \left( 1 + \alpha_m_3 r_{o1} \right)}\]

\[V_{x} = \frac{1}{r_{o2}} + \frac{\alpha_m - \alpha_m_3}{1 + \frac{1}{r_{o3}}} \cdot \frac{V_{out}}{r_{o1}} \]

\[V_{out} = \frac{1}{r_{o2}} + \frac{\alpha_m}{\alpha_m_3} \cdot \frac{V_{out} - \alpha_m_3 V_{out}}{r_{o1}} \]
1. \[ I_{D1} = I_{D2} \Rightarrow V_{in} = 1.254 \Rightarrow I_D = 2.316 \text{ mA} \]

\[ g_m1 = I_n C_m \left( \frac{W}{L} \right), \quad (V_{GS1} - V_{TH}) = 1.34 \times 10^{-4} \times 100 \times (1.254 - 0.7) = 7.43 \text{ mV} \]

\[ g_m2 = \frac{I_p C_m \left( \frac{W}{L} \right)^2 \left( V_{GS2} - V_{TH}^p \right)}{3.84 \times 10^{-5} \times 100 (3 - 1.254 - 0.8) = 3.63 \text{ mV} \]

\[ r_01 = \frac{1}{\lambda_1 I_1} = 4.317 \text{ k} \quad \quad r_{o2} = \frac{1}{\lambda_2 I_2} = 2.159 \text{ k} \]

(a) \[ a) \quad A_V = -(g_{m1} + g_{m2}) \frac{r_{o1} r_{o2}}{r_{o1} + r_{o2}} = 15.91 \]

\[ R_{out} = r_{o1} r_{o2} = 14.39 \text{ k} \]

(b) \[ V_g = V_{in} x \frac{R_2}{R_1 + R_2} + V_o \frac{R_1}{R_1 + R_2} \]

\[ \frac{V_o - V_{in}}{R_1 + R_2} + \frac{V_o}{r_{o1} r_{o2}} + (g_{m1} + g_{m2}) \left[ \frac{V_{in} R_2}{R_1 + R_2} + \frac{V_o R_1}{R_1 + R_2} \right] = 0 \]

\[ \Rightarrow A_V = \frac{V_o}{V_{in}} = -\frac{(g_{m1} + g_{m2}) R_2 - 1}{1 + (g_{m1} + g_{m2}) R_1 \frac{R_1 + R_2}{r_{o1} r_{o2}}} = 5.57 \]

To calculate \( R_{out} \):

\[ V_{in} = 0 \]

\[ i_o = \frac{V_o}{r_{o1} r_{o2}} + \frac{V_o}{R_1 + R_2} + (g_{m1} + g_{m2}) \left[ \frac{V_o R_2}{R_1 + R_2} \right] \]

\[ = V_o \left[ \frac{1}{r_{o1} r_{o2}} + \frac{1 + (g_{m1} + g_{m2}) R_1}{R_1 + R_2} \right] \]

\[ \Rightarrow R_{out} = \frac{V_o}{i_o} = 557 \text{ k} \]

(b) We figure out sensitivity (for (b)),

(a) is a special case where \( R_1 = \infty \) and \( R_2 = \infty \)
\[ V_g = V_o \times \frac{R_1}{R_1 + R_2} \]

\[
\frac{V_o - V_{dd}}{R_{o2}} + \frac{V_o}{R_{o1}} + \frac{V_o}{R_1 + R_2} + g_{m1} \times V_o \times \frac{R_1}{R_1 + R_2} + g_{m2} \left( V_o \times \frac{R_1}{R_1 + R_2} - V_{dd} \right) = 0
\]

\[ \Rightarrow \frac{g_{m2} + \frac{1}{R_{o2}}}{\frac{1}{R_{o1}} + \frac{1}{R_{o2}} + \frac{1}{R_1 + R_2} + \left( g_{m1} + g_{m2} \right) \frac{R_1}{R_1 + R_2}} = 2.28 = A_v \]

If \( R_1 = 0 \) and \( R_2 = \infty \)

\[ A_v = \left[ g_{m2} + \frac{1}{R_{o2}} \right] \left( R_{o1} || R_{o2} \right) = (3.63 + 0.46) \times 1.439 \]

\[ = 5.84 \]