Noise Variance in a Real Circuit: Sample and Hold

- Noise on the capacitor:
  \[ v_{\text{on}}(f) = 4k_BTR \left| \frac{1}{1+sRC} \right|^2 \]
  \[ \Rightarrow \bar{v}_{\text{on}}^2 = \int_0^\infty v_{\text{on}}^2(f)df = \frac{k_B}{C} \]

- So effective bandwidth is:
  \[ 4k_BTR\Delta f = \frac{k_B}{C} \]
  \[ \Rightarrow \Delta f = \frac{1}{4RC} = \frac{\pi}{2}f_o \]
SPICE Verification

Useful Integrals

\[ \int_0^\infty \frac{1}{1 + \frac{s^2}{\omega^2}} \, df = \frac{\omega}{4} \]

\[ \int_0^\infty \frac{1}{s + \frac{s^3}{\omega Q} + \frac{s^2}{\omega^2}} \, df = \int_1^\infty \frac{s}{s + \frac{s^3}{\omega Q} + \frac{s^2}{\omega^2}} \, df = \frac{\omega Q}{4} \]

\[ \int_0^\infty \frac{s + 1}{s + \frac{s^3}{\omega Q} + \frac{s^2}{\omega^2}} \, df = \frac{\omega Q}{4} \left( \frac{\omega^2}{\omega^2 + 1} \right) \]
SC Resistor Noise Spectrum

\[ S_y(f) = \frac{kT_s}{C} \sqrt{\frac{1}{2 \pi f_s} \left( 1 - e^{-rf_s} \right)} \]

- Noise essentially white for \( T/\tau > 3 \)
- Setting constraints ensure that this condition is usually met in practice
- Note: This is the noise density of an SC resistor only. The noise density from an SC filter is usually not white.

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Periodic Noise Analysis

Sampling Noise from SC S/H

SpectreRF PNOISE: check
noisetype=timedomain
noisetpoint=[...]

as alternative to ZOH.
noiseskipcount=large
might speed up things in this case.
Another Example: CS Amplifier

\[ v_{m}^{\prime}(f) = 4k_{B}T \left( \frac{1}{R_{L}} + \frac{2}{3} g_{m} \right) R_{L} \left[ \frac{1}{1 + sR_{L}C_{L}} \right] \]

\[ v_{st}^{\prime} = 4k_{B}T \left( \frac{1}{R_{L}} + \frac{2}{3} g_{m} \right) R_{L}^{2} \left[ \frac{1}{1 + sR_{L}C_{L}} \right] d/f \]

\[ = 4k_{B}T \left( \frac{1}{R_{L}} + \frac{2}{3} g_{m} R_{L} \right) \]

\[ = k_{B}T \left( 1 + \frac{2}{3} g_{m} R_{L} \right) \]

\[ = k_{B}T \left( 1 + \frac{2}{3} |A_{m}| \right) \]

\[ = \frac{k_{B}T}{C_{L}} n_{f} \]

Signal-To-Noise Ratio

- **SNR:**

\[ SNR = \frac{P_{\text{sig}}}{P_{\text{noise}}} \]

- **Signal Power (sinusoidal source):**

\[ P_{\text{sig}} = \frac{1}{2} V_{\text{zero--peak}}^{2} \]

- **Noise Power (assuming thermal noise dominates):**

\[ P_{\text{noise}} = \frac{k_{B}T}{C} n_{f} \]

- **So:**

\[ SNR = \frac{1}{2} CV_{\text{zero--peak}}^{2} n_{f} k_{B}T \]

\[ SNR \quad \uparrow +6dB \]

\[ \downarrow \times 4 \]
**dB versus Bits**

- **Quantization “noise”**
  - Quantizer step size: \( \Delta \)
  - Box-car pdf variance: \( S_Q = \frac{\Delta^2}{12} \)

- **SNR of N-Bit sinusoidal signal**
  - Signal power
    \[
    P_{\text{sig}} = \frac{1}{2} \left( 2^N \frac{\Delta}{2} \right)^2
    \]
  - SNR
    \[
    \text{SNR} = \frac{P_{\text{sig}}}{S_Q} = 1.5 \times 2^N
    \]
  - 6.02 dB per Bit
    \[
    = \left[ 1.76 + 6.02N \right] \text{ dB}
    \]

<table>
<thead>
<tr>
<th>N</th>
<th>dB</th>
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<tbody>
<tr>
<td>8</td>
<td>50</td>
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<tr>
<td>16</td>
<td>98</td>
</tr>
<tr>
<td>24</td>
<td>146</td>
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</table>

**SNR versus Power**

- 1 Bit \( \rightarrow \) 6dB \( \rightarrow \) 4x SNR
- 4x SNR \( \rightarrow \) 4x C
- Circuit bandwidth \( \sim g_m / C \rightarrow 4x g_m \)
- Keeping \( V^* \) constant \( \rightarrow 4x I_b, 4x W \)

- Thermal noise limited circuit:
  - Each bit QUADRUPLES power!

- Overdesign is **expensive**
  - Better do the analysis right!
Analog Circuit Dynamic Range

- Biggest signal set by $V_{DD}$: So, for (single-ended) sinusoid:
  \[
  V_{\text{max (rms)}} = \frac{1}{\sqrt{2}} \frac{V_{DD}}{2}
  \]

- The noise is
  \[
  V_n(rms) = \sqrt{n_f \frac{k_B T}{C}}
  \]

- So the dynamic range in dB is:
  \[
  DR = \frac{V_{\text{max (rms)}}}{V_n(rms)} = \frac{V_{DD} \sqrt{C}}{\sqrt{8n_f k_B T}} \quad [V/V]
  \]
  \[
  = 20 \log_{10} \left( V_{DD} \frac{C}{n_f} \right) + 75 \quad [\text{dB}] \text{ with } C \text{ in } [\text{pF}]
  \]
Input Equivalent Noise

Equivalent Noise Generators

- Model for noisy two-port:
  - *Noiseless* two-port
  - Plus equivalent input noise sources

- In general, $v_n$ and $i_n$ are correlated.
  - Ignore that for now
Finding the Equivalent Generators

- Find $v_n$ and $i_n$ by opening and shorting the input
  - Shorted input:
    - Output noise due only to $v_n$
  - Open input:
    - Output noise due only to $i_n$

Role of Source Resistance

- If $R_s$ is large:
  - Design amplifier with low $i_n$ (MOS)
- If $R_s$ is low:
  - Design amplifier with low $v_n$ (BJT)

- For a given $R_s$, there is an optimal $v_n/i_n$ ratio
  - Alternatively, for a given amp, there is an optimal $R_s$
Total Output Noise

\[
\bar{v}_n^2 = \left( \bar{v}_n^2 A_n^2 + \bar{v}_{Rs}^2 A_n^2 \right) \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 + \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 R_s^2 \bar{v}_n^2 A_n^2
\]

\[
= \left( \bar{v}_n^2 + \bar{v}_{Rs}^2 \right) \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 A_n^2
\]

New Equivalent Generator

\[
\bar{v}_{eq}^2 = \bar{v}_n^2 + \bar{v}_{Rs}^2 R_s^2
\]

- With known \( R_s \), total noise can be lumped into one \( \bar{v}_{eq} \)
**Optimum Source Impedance**

- Can use this to optimize source impedance for minimum added noise from two-port (noise figure):

\[
R_n = \frac{v_n^2}{4kT\Delta f} \quad G_n = \frac{i_n^2}{4kT\Delta f} \quad R_{opt} = \sqrt{\frac{R_n}{G_n}} = \sqrt{\frac{v_n^2}{i_n^2}}
\]

**Correlated Noise Sources**

- Partition into two components:
  - Correlated ("parallel") to \( v_n \)
  - Uncorrelated ("perpendicular") to \( v_n \)

\[
i_n = i_c + i_u \quad <i_u, v_n> = 0 \quad i_c = Y_C v_n
\]

- Partition \( i_n \) into two components:
  - Correlated ("parallel") to \( v_n \)
  - Uncorrelated ("perpendicular") to \( v_n \)
Correlated Noise Sources (cont.)

Finding $Y_c$:

\[
\overline{v_{eq}^2} = \overline{v_n^2} + Z_s^2 \overline{i_n^2} \\
\overline{v_n^2} = \overline{v_n^2} + Z_s^2 \overline{(i_c + i_u)^2} \\
\overline{v_n^2} = \overline{v_n^2} + Z_s^2 \overline{i_c^2} + Z_s^2 \overline{i_u^2} \\
\overline{v_n^2} = \overline{v_n^2} (1 + Z_s^2 Y_c^2) + Z_s^2 \overline{i_u^2} \\
\overline{i_n^2} = \overline{i_c^2} + \overline{i_u^2} \\
\overline{i_c^2} = \beta_2^2 \overline{Y_c^2} + \beta_3^2 \overline{v_3^2} \\
Y_c^2 = \frac{\overline{i_c^2}}{\overline{v_n^2}} \\
\alpha_1^2 \overline{v_1^2} + \alpha_2^2 \overline{i_2^2}
\]

Equivalent Noise Voltage (cor)

- Since the above expression is the sum of two uncorrelated noise voltages, we have

\[
\overline{v_{eq}^2} = \overline{v_n^2} |1 + Y_C Z_S|^2 + |Z_S|^2 \overline{i_u^2}
\]

- Now we can continue as before to find

\[
B_{opt} = B_s = -B_c \\
G_{opt} = G_s = \sqrt{\frac{C_u}{R_n} + G_c^2}
\]