Chapter 7

Resonance and Impedance Matching

Many common circuits make use of inductors and capacitors in different ways to achieve their functionality. Filters, impedance matching circuits, resonators, and chokes are common examples. We study these circuits in detail and in particular we shall focus on the desirable properties of the passive components in such circuits.

7.1 Resonance

We begin with the textbook discussion of resonance of *RLC* circuits. These circuits are simple enough to allow full analysis, and yet rich enough to form the basis for most of the circuits we will study in this chapter.

Incidentally simple second-order resonant circuit can also model a wide array of physical phenomena, such as pendulums, mass-spring mechanical resonators, molecular resonance, microwave cavities, sections of transmission lines, and even large scale structures such as bridges. Understanding these circuits will afford a wide perspective into many physical situations.

Series RLC Circuits

The *RLC* circuit shown in Fig. 7.1 is deceptively simple. The impedance seen by the source is simply given by

$$Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right) \tag{7.1}$$

The impedance is purely real at at the *resonant frequency* when $\Im(Z) = 0$, or $\omega = \pm\frac{1}{\sqrt{LC}}$. At resonance the impedance takes on a minimal value. It's worthwhile to investigate the cause of resonance, or the cancellation of the reactive components due to the inductor and

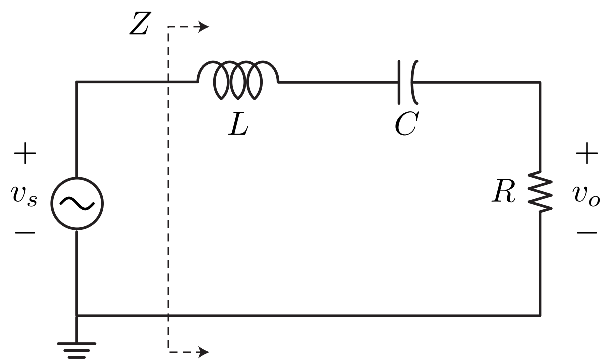


Figure 7.1: A series RLC circuit.

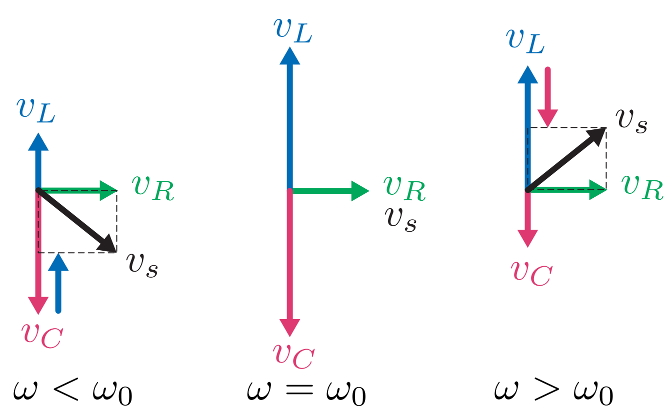


Figure 7.2: The phasor diagram of voltages in the series RLC circuit (a) below resonance, (b) at resonance, and (c) beyond resonance.

capacitor. Since the inductor and capacitor voltages are always 180° out of phase, and one reactance is dropping while the other is increasing, there is clearly always a frequency when the magnitudes are equal. Thus resonance occurs when $\omega L = \frac{1}{\omega C}$. A phasor diagram, shown in Fig. 7.2, shows this in detail.

So what's the magic about this circuit? The first observation is that at resonance, the voltage across the reactances can be larger, in fact much larger, than the voltage across the resistors R. In other words, this circuit has voltage gain. Of course it does not have power gain, for it is a passive circuit. The voltage across the inductor is given by

$$v_L = j\omega_0 L i = j\omega_0 L \frac{v_s}{Z(j\omega_0)} = j\omega_0 L \frac{v_s}{R} = jQ \times v_s \tag{7.2}$$

where we have defined a circuit Q factor at resonance as

$$Q = \frac{\omega_0 L}{R} \tag{7.3}$$

It's easy to show that the same voltage multiplication occurs across the capacitor

$$v_C = \frac{1}{j\omega_0 C} i = \frac{1}{j\omega_0 C} \frac{v_s}{Z(j\omega_0)} = \frac{1}{j\omega_0 C} \frac{v_s}{R} = -jQ \times v_s \tag{7.4}$$

This voltage multiplication property is the key feature of the circuit that allows it to be used as an impedance transformer.

It's important to distinguish this Q factor from the intrinsic Q of the inductor and capacitor. For now, we assume the inductor and capacitor are ideal. We can re-write the Q factor in several equivalent forms owing to the equality of the reactances at resonance

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C} \frac{1}{R} = \frac{\sqrt{LC}}{C} \frac{1}{R} = \sqrt{\frac{L}{C}} \frac{1}{R} = \frac{Z_0}{R} \tag{7.5}$$

where we have defined the $Z_0 = \sqrt{\frac{L}{C}}$ as the characteristic impedance of the circuit.

Circuit Transfer Function

Let's now examine the transfer function of the circuit

$$H(j\omega) = \frac{v_o}{v_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} \tag{7.6}$$

$$H(j\omega) = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \tag{7.7}$$

Obviously, the circuit cannot conduct DC current, so there is a zero in the transfer function. The denominator is a quadratic polynomial. It's worthwhile to put it into a standard form that quickly reveals important circuit parameters

$$H(j\omega) = \frac{j\omega \frac{R}{L}}{\frac{1}{LC} + (j\omega)^2 + j\omega \frac{R}{L}} \tag{7.8}$$

Using the definition of Q and ω_0 for the circuit

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}} \tag{7.9}$$

Factoring the denominator with the assumption that $Q > \frac{1}{2}$ gives us the complex poles of the circuit

$$s^{\pm} = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1 - \frac{1}{4Q^2}} \tag{7.10}$$

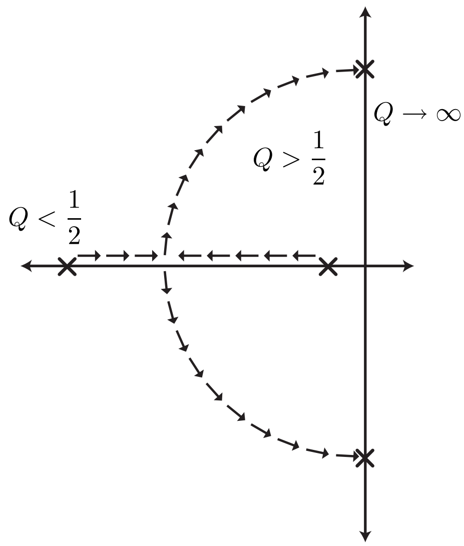


Figure 7.3: The root locus of the poles for a second-order transfer function as a function of Q. The poles begin on the real axis for $Q < \frac{1}{2}$ and become complex, tracing a semi-circle for increasing Q.

The poles have a constant magnitude equal to the resonant frequency

$$|s| = \sqrt{\frac{\omega_0^2}{4Q^2} + \omega_0^2\left(1 - \frac{1}{4Q^2}\right)} = \omega_0 \tag{7.11}$$

A root-locus plot of the poles as a function of Q appears in Fig. 7.3. As $Q \to \infty$, the poles move to the imaginary axis. In fact, the real part of the poles is inversely related to the Q factor.

Circuit Bandwidth

As shown in Fig. 7.4, when we plot the magnitude of the transfer function, we see that the selectivity of the circuit is also related inversely to the Q factor. In the limit that $Q \to \infty$, the circuit is infinitely selective and only allows signals at resonance ω_0 to travel to the load. Note that the peak gain in the circuit is always unity, regardless of Q, since at resonance the L and C together disappear and effectively all the source voltage appears across the load.

The selectivity of the circuit lends itself well to filter applications (Sec. 7.5). To characterize the peakiness, let's compute the frequency when the magnitude squared of the transfer

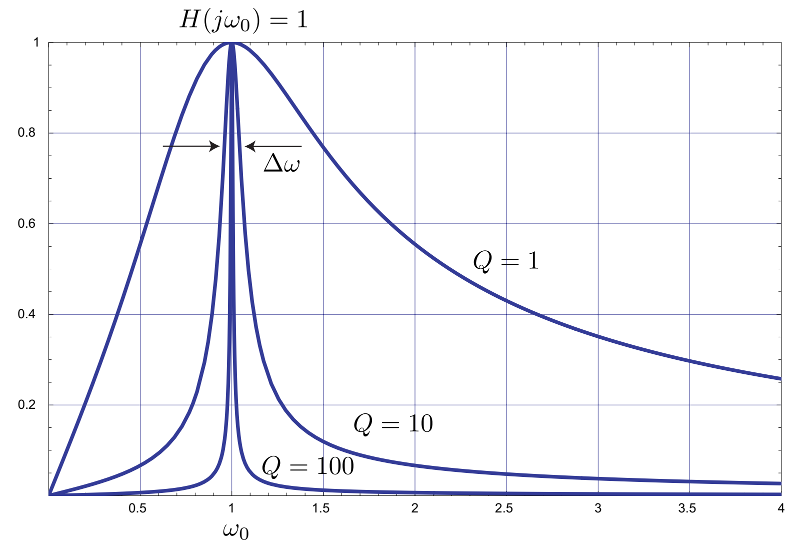


Figure 7.4: The transfer function of a series RLC circuit. The output voltage is taken at the resistor terminals. Increasing Q leads to a more peaky response.

function drops by half

$$|H(j\omega)|^2 = \frac{\left(\omega\frac{\omega_0}{Q}\right)^2}{(\omega_0^2 - \omega^2)^2 + \left(\omega\frac{\omega_0}{Q}\right)^2} = \frac{1}{2} \tag{7.12}$$

This happens when

$$\left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right)^2 = 1 \tag{7.13}$$

Solving the above equation yields four solutions, corresponding to two positive and two negative frequencies. The peakiness is characterized by the difference between these frequencies, or the bandwidth, given by

$$\Delta\omega = \omega_+ - \omega_- = \frac{\omega_0}{Q} \tag{7.14}$$

which shows that the normalized bandwidth is inversely proportional to the circuit Q

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q} \tag{7.15}$$

You can also show that the resonance frequency is the geometric mean frequency of the 3 dB frequencies

$$\omega_0 = \sqrt{\omega_+\omega_-} \tag{7.16}$$

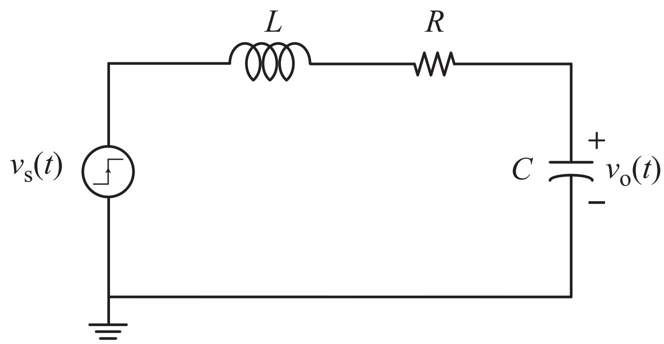


Figure 7.5: A step function applied to a series RLC circuit with output taken across the capacitor.

Circuit Damping Factor

So far we have characterized the second order series RLC circuit by its frequency response, leading to the concept of the resonant frequency ω_0 and quality factor Q. An equivalent characterization involves the time domain response, particularly the step response of the circuit. This shall lead to the concept of the damping factor. As shown in Fig. 7.5, we shall now consider the output as the voltage across the capacitor. This corresponds to the common situation in digital circuits, where the "load" is a capacitive reactance of a gate.

First let's recall the familiar step response of an RC circuit, or the limit as $L \to 0$ in the series RLC circuit, the circuit response to a step function is a rising exponential function that asymptotes towards the source voltage with a time scale $\tau = 1/RC$.

When the circuit has inductance, it's described by a second order differential equation. Applying KVL we can write an equation for the voltage $v_C(t)$

$$v_s(t) = v_C(t) + RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} \tag{7.17}$$

with initial conditions

$$v_0(t) = v_C(t) = 0\text{V} \tag{7.18}$$

and

$$i(0) = i_L(0) = 0\text{A} \tag{7.19}$$

For $t > 0$, the source voltage switches to V_{dd}. Thus Eq. 7.17 has a constant source for $t > 0$

$$V_{dd} = v_C(t) + RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} \tag{7.20}$$

In steady-state, $\frac{d}{dt} \to 0$, leading to

$$V_{dd} = v_C(\infty) \tag{7.21}$$

which implies that the entire voltage of the source appears across the capacitor, driving the current in the circuit to zero. Subtracting this steady-state voltage from the solution, we can simplify our differential equation

$$v_C(t) = V_{dd} + v(t) \tag{7.22}$$

$$V_{dd} = V_{dd} + v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} \tag{7.23}$$

or more simply, the homogeneous equation

$$0 = v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} \tag{7.24}$$

The solution of which is nothing but a complex exponential $v(t) = Ae^{st}$, leading to the characteristic equation

$$0 = 1 + RCs + LCs^2 = 1 + (s\tau)2\zeta + (s\tau)^2 \tag{7.25}$$

where we have defined

$$\tau = \frac{1}{\omega_0} \tag{7.26}$$

and

$$\zeta = \frac{1}{2Q} \tag{7.27}$$

which has solutions

$$s\tau = -\zeta \pm \sqrt{\zeta^2 - 1} \tag{7.28}$$

This equation can be characterized by the value ζ, leading to the following three important cases

$$\begin{array}{ll} \zeta < 1 & \text{Underdamped} \\ \zeta = 1 & \text{Critically Damped} \\ \zeta > 1 & \text{Overdamped} \end{array} \tag{7.29}$$

The terminology "damped" will become obvious momentarily. When we plot the general time-domain solution, we have

$$v_C(t) = V_{dd} + A\exp(s_1 t) + B\exp(s_2 t) \tag{7.30}$$

which must satisfy the following boundary conditions

$$v_C(0) = V_{dd} + A + B = 0 \tag{7.31}$$

and

$$i(0) = C\frac{dv_C(t)}{dt}|_{t=0} = 0 \tag{7.32}$$

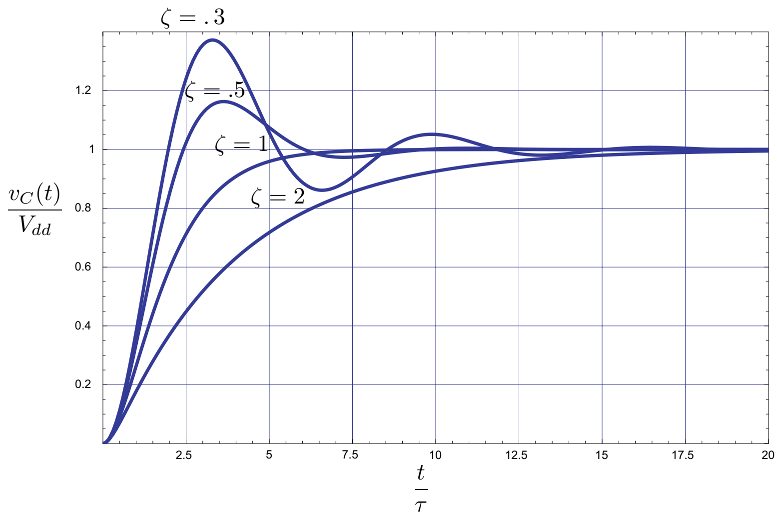


Figure 7.6: The normalized step response of an RLC circuit. The damping factor ζ is varied to cover three important regions, an overdamped response with $\zeta > 1$, a critically damped response with $\zeta = 1$, and an underdamped response with $\zeta < 1$.

or the following set of equations

$$As_1 + Bs_2 = 0 \tag{7.33}$$

$$A + B = -V_{dd} \tag{7.34}$$

which has the following solution

$$A = \frac{-V_{dd}}{1 - \sigma} \tag{7.35}$$

$$B = \frac{\sigma V_{dd}}{1 - \sigma} \tag{7.36}$$

where $\sigma = \frac{s_1}{s_2}$. So we have the following equation

$$v_C(t) = V_{dd}\left(1 - \frac{1}{1 - \sigma}\left(e^{s_1 t} - \sigma e^{s_2 t}\right)\right) \tag{7.37}$$

If the circuit is overdamped, $\zeta > 1$, the poles are real and negative

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 \\ s_2 \end{cases} < 0 \tag{7.38}$$

The normalized response, with $\zeta = 2, 1, 0.5, 0.3$, is shown in Fig. 7.6. Qualitatively we see that the $\zeta = 2$ circuit response resembles the familiar RC circuit. This is true until

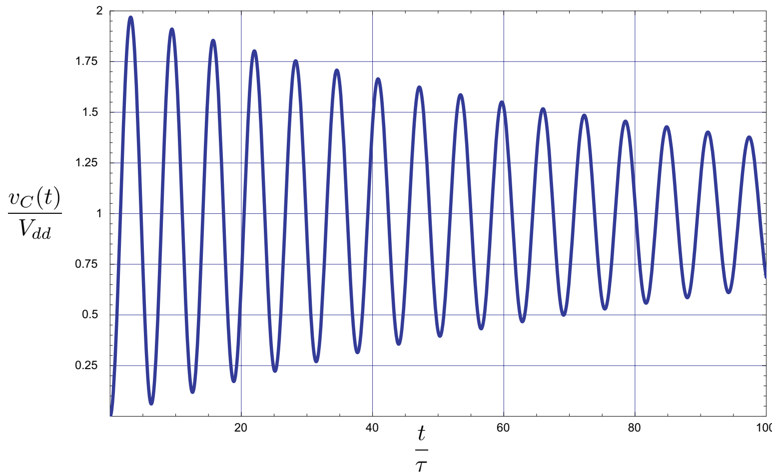


Figure 7.7: The step response of an RLC circuit with low damping ($\zeta = .01$).

$\zeta = 1$, at which point the circuit is said to be *critically damped*. The circuit has two equal poles at

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau} \tag{7.39}$$

In which case the time response is given by

$$\lim_{\zeta \to 1} v_C(t) = V_{dd}\left(1 - e^{-t/\tau} - \frac{t}{\tau}e^{-t/\tau}\right) \tag{7.40}$$

As seen in Fig. 7.6, the circuit step response is faster with increasing $\zeta \leq 1$, but still similar to the RC circuit. The *underdamped* case, $\zeta < 1$, is the most interesting case here, as the response is now markedly different. The circuit *overshoots* the mark and settles down to the steady-state value after oscillating. The amount of oscillation increases as the damping factor ζ is reduced. The response with very low damping, $\zeta = .01$ is shown in Fig. 7.7.

The underdamped case is characterized by two complex conjugate poles

$$s\tau = -\zeta \pm j\sqrt{1 - \zeta^2} = a \pm jb \tag{7.41}$$

Note the complex conjugate of the A coefficient is simply

$$A^* = \frac{-V_{dd}}{1 - \sigma^*} = \frac{V_{dd}}{\sigma^{-1} - 1} = \frac{\sigma V_{dd}}{1 - \sigma} = B \tag{7.42}$$

which allows us to write the voltage across the capacitor as

$$v_C(t) = V_{dd} + e^{at/\tau}\left(Ae^{jbt\tau} + A^*e^{-jbt\tau}\right) \tag{7.43}$$

or more simply

$$v_C(t) = V_{dd} + e^{at/\tau} 2|A| \cos(\omega t + \phi) \tag{7.44}$$

where

$$|A| = \frac{V_{dd}}{|1+\sigma|} \tag{7.45}$$

and

$$\phi = \angle \frac{V_{dd}}{1+\sigma} \tag{7.46}$$

The oscillating frequency is determined by the imaginary part of the poles, $\omega = b/\tau = \omega_0\sqrt{1-\zeta^2}$, which approaches the natural resonance frequency as the damping is reduced. The decay per period, or the envelope damping of the waveform, is determined by a, which equals $\zeta\omega_0$. This is why the damping factor controls the amount of overshoot and oscillation in the waveform.

Energy Storage in *RLC* "Tank"

Let's compute the ebb and flow of the energy at resonance. To begin, let's assume that there is negligible loss in the circuit. The energy across the inductor is given by

$$w_L = \tfrac{1}{2}Li^2(t) = \tfrac{1}{2}LI_M^2 \cos^2 \omega_0 t \tag{7.47}$$

Likewise, the energy stored in the capacitor is given by

$$w_C = \tfrac{1}{2}Cv_C^2(t) = \tfrac{1}{2}C\left(\frac{1}{C}\int i(\tau)d\tau\right)^2 \tag{7.48}$$

Performing the integral leads to

$$w_C = \tfrac{1}{2}\frac{I_M^2}{\omega_0^2 C}\sin^2 \omega_0 t \tag{7.49}$$

The total energy *stored* in the circuit is the sum of these terms

$$w_s = w_L + w_C = \tfrac{1}{2}I_M^2\left(L\cos^2\omega_0 t + \frac{1}{\omega_0^2 C}\sin^2\omega_0 t\right) = \tfrac{1}{2}I_M^2 L \tag{7.50}$$

which is a constant! This means that the reactive stored energy in the circuit does not change and simply moves between capacitive energy and inductive energy. When the current is maximum across the inductor, all the energy is in fact stored in the inductor

$$w_{L,\max} = w_s = \tfrac{1}{2}I_M^2 L \tag{7.51}$$

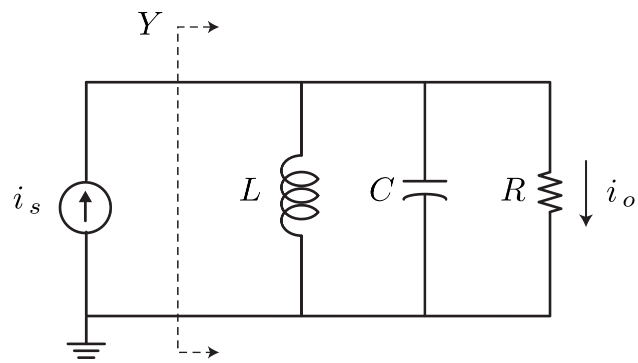


Figure 7.8: A parallel RLC circuit.

Likewise, the peak energy in the capacitor occurs when the current in the circuit drops to zero

$$w_{C,\max} = w_s = \tfrac{1}{2}V_M^2 C \tag{7.52}$$

Now let's re-introduce loss in the circuit. In each cycle, a resistor R will dissipate energy

$$w_d = P \cdot T = \tfrac{1}{2}I_M^2 R \cdot \frac{2\pi}{\omega_0} \tag{7.53}$$

The ratio of the energy stored to the energy dissipated is thus

$$\frac{w_s}{w_d} = \frac{\frac{1}{2}LI_M^2}{\frac{1}{2}I_M^2 R \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R}\frac{1}{2\pi} = \frac{Q}{2\pi} \tag{7.54}$$

This gives us the physical interpretation of the *Quality Factor* Q as 2π times the ratio of energy stored per cycle to energy dissipated per cycle in an RLC circuit

$$Q = 2\pi \frac{w_s}{w_d} \tag{7.55}$$

We can now see that if $Q \gg 1$, then an initial energy in the tank tends to slosh back and forth for many cycles. In fact, we can see that in roughly Q cycles, the energy of the tank is depleted.

Parallel RLC Circuits

The parallel RLC circuit shown in Fig. 7.8 is the dual of the series circuit. By "dual" we mean that the role of voltage and current are interchanged. Hence the circuit is most naturally probed with a current source i_s. In other words, the circuit has current gain as opposed to voltage gain, and the admittance minimizes at resonance as opposed to the impedance.

Finally, the role of capacitance and inductance are also interchanged. In principle, therefore, we don't have to repeat all the detailed calculations we just performed for the series case, but in practice it's worthwhile exercise.

The admittance of the circuit is given by

$$Y = j\omega C + \frac{1}{j\omega L} + G = G + j\omega C\left(1 - \frac{1}{\omega^2 LC}\right) \tag{7.56}$$

which has the same form as Eq. 7.1. The resonant frequency also occurs when $\Im(Y) = 0$, or when $\omega = \omega_0 = \pm\frac{1}{\sqrt{LC}}$. Likewise, at resonance the admittance takes on a minimal value. Equivalently, the impedance at resonance is maximum. This property makes the parallel RLC circuit an important element in tuned amplifier loads (see Sec. 8). It's also easy to show that at resonance the circuit has a current gain of Q

$$i_C = j\omega_0 C v_o = j\omega_0 C\frac{i_s}{Y(j\omega_0)} = j\omega_0 C\frac{i_s}{G} = jQ \times i_s \tag{7.57}$$

where we have defined the circuit Q factor at resonance by

$$Q = \frac{\omega_0 C}{G} \tag{7.58}$$

in complete analogy with Eq. 7.3. Likewise, the current gain through the inductor is also easily derived

$$i_L = -jQ \times i_s \tag{7.59}$$

The equivalent expressions for the circuit Q factor are given by the inverse of the relations of Eq. 7.5

$$Q = \frac{\omega_0 C}{G} = \frac{R}{\omega_0 L} = \frac{R}{\frac{1}{\sqrt{LC}}L} = \frac{R}{\sqrt{\frac{L}{C}}} = \frac{R}{Z_0} \tag{7.60}$$

The phase response of a resonant circuit is also related to the Q factor. For the parallel RLC circuit the phase of the admittance is given by

$$\angle Y(j\omega) = \tan^{-1}\left(\frac{\omega C\left(1 - \frac{1}{\omega^2 LC}\right)}{G}\right) \tag{7.61}$$

The rate of change of phase at resonance is given by

$$\left.\frac{d\angle Y(j\omega)}{d\omega}\right|_{\omega_0} = \frac{2Q}{\omega_0} \tag{7.62}$$

A plot of the admittance phase as a function of frequency and Q is shown in Fig. 7.9.

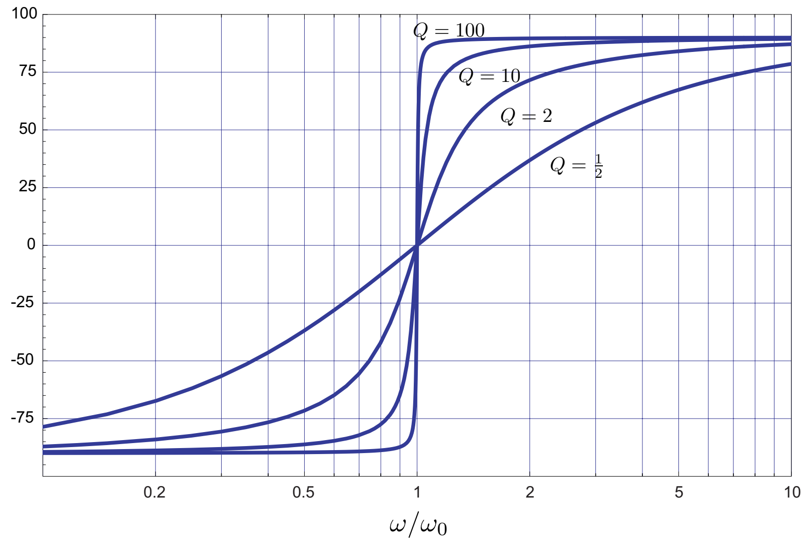


Figure 7.9: The phase of a second order admittance as function of frequency. The rate of change of phase at resonance is proportional to the Q factor.

Circuit Transfer Function

Given the duality of the series and parallel RLC circuits, it's easy to deduce the behavior of the circuit. Whereas the series RLC circuit acted as a filter and was only sensitive to voltages near resonance ω_0, likewise the parallel RLC circuit is only sensitive to currents near resonance

$$H(j\omega) = \frac{i_o}{i_s} = \frac{v_o G}{v_o Y(j\omega)} = \frac{G}{j\omega C + \frac{1}{j\omega L} + G} \tag{7.63}$$

which can be put into the same canonical form as before

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}} \tag{7.64}$$

where we have appropriately re-defined the circuit Q to correspond the parallel RLC circuit. Notice that the impedance of the circuit takes on the same form

$$Z(j\omega) = \frac{1}{Y(j\omega)} = \frac{1}{j\omega C + \frac{1}{j\omega L} + G} \tag{7.65}$$

which can be simplified to

$$Z(j\omega) = \frac{j\frac{\omega}{\omega_0}\frac{1}{GQ}}{1 + \left(\frac{j\omega}{\omega_0}\right)^2 + j\frac{\omega}{\omega_0 Q}} \tag{7.66}$$

At resonance, the real terms in the denominator cancel

$$Z(j\omega_0) = \frac{j\frac{R}{Q}}{\underbrace{1 + \left(\frac{j\omega_0}{\omega_0}\right)^2}_{=0} + j\frac{1}{Q}} = R \tag{7.67}$$

It's not hard to see that this circuit has the same half power bandwidth as the series RLC circuit, since the denominator has the same functional form

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q} \tag{7.68}$$

A plot of this impedance versus frequency has the same form as Fig. 7.4 multiplied by the resistance R.

Energy storage in a parallel RLC circuit is completely analogous to the series RLC case and in fact the general equation relating circuit Q to energy storage and dissipation also holds in the parallel RLC circuit.

7.2 The Many Faces of Q

As we have seen, in RLC circuits the most important parameter is the circuit Q and resonance frequency ω_0. Not only do these parameters describe the circuit in a general way, but they also give us immediate insight into the circuit behavior.

The Q factor can be computed several ways, depending on the application. For instance, if the circuit is designed as a filter, then the most important Q relation is the half-power bandwidth

$$Q = \frac{\omega_0}{\Delta\omega} \tag{7.69}$$

We shall also find many applications where the phase selectivity of these circuits is of importance. An example is a resonant oscillator where the noise of the system is rejected by the tank based on the phase selectivity. In an oscillator any "excess phase" in the loop tends to move the oscillator away from the natural resonant frequency. It is therefore desirable to maximize the rate of change of phase of the circuit impedance as a function of frequency. For the parallel RLC circuit we derived the phase of the admittance (Eq. 7.62) which gives us another way to interpret and compute Q

$$Q = \frac{\omega_0}{2}\frac{d\angle Y(j\omega)}{d\omega} \tag{7.70}$$

For applications where the circuit is used as a voltage or current multiplier, the ratio of reactive voltage (current) to real voltage (current) is most relevant. As we shall see, in

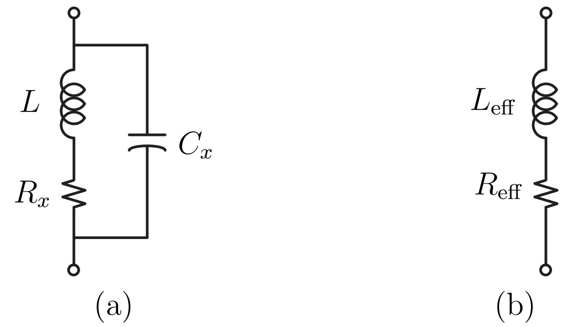


Figure 7.10: (a) A simple model for an inductor with winding capacitance C_x and winding resistance R_x. (b) A simplified equivalent circuit for the non-ideal inductor model.

RFID systems (Sec. 10.10) this is an important application of the circuit. For a series case we found

$$Q = \frac{v_L}{v_R} = \frac{v_C}{v_R} \tag{7.71}$$

and for the parallel case

$$Q = \frac{i_L}{i_R} = \frac{i_C}{i_R} \tag{7.72}$$

Finally, when the step response or time domain transient response is of interest, the circuit Q or equivalently, the damping factor ζ, describes the behavior of the circuit, with $\zeta = 1$ the boundary between damped behavior and under-damped oscillatory response.

The last and one of the most important interpretations of Q is in the definition of energy, relating the energy storage and losses in a RLC "tank" circuit. We can define the of Q a circuit at frequency ω as the energy stored in the tank W divided by the rate of energy loss

$$Q = W/\frac{dW}{d\phi} = \omega W/\frac{dW}{dt}$$

Practical Issues with Resonators

Inductor Equivalent Circuit

The previous discussion was oversimplified since any real RLC circuit will consists of non-ideal and lossy L's and C's. For example, consider the simple equivalent circuit for the inductor shown in Fig. 7.10a, where C_x accounts for the winding capacitance and R_x is the series winding resistance. This simple model is sufficiently accurate over a narrow range of frequencies for inductors realized on non-lossy substrates or an air coil inductor. In many situations we'd like to model the inductor as an equivalent ideal inductor with series loss, as shown in Fig. 7.10b. This is done by simply equating the impedance at a given frequency

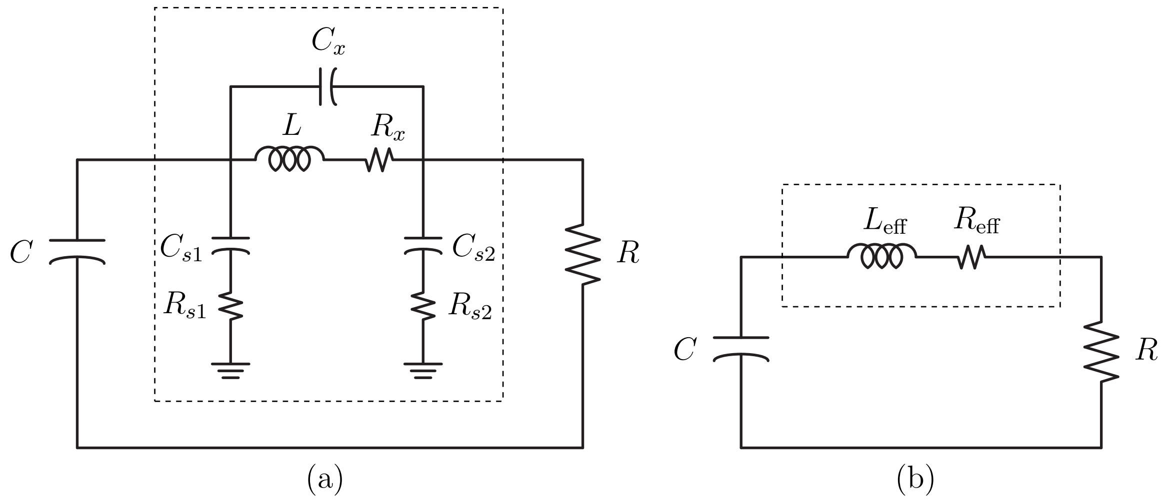


Figure 7.11: (a) A series RLC circuit containing a non-ideal inductor L. (b) A simplified equivalent circuit for the series RLC circuit.

resulting in

$$R_{\text{eff}} \approx R\left(1+\left(\frac{\omega}{\omega_0}\right)^2\right) \tag{7.73}$$

$$L_{\text{eff}} \approx L\left(1+\left(\frac{\omega}{\omega_0}\right)^2\right) \tag{7.74}$$

The above approximation holds when the circuit is operated far below the inductor self-resonant frequency (SRF) $\omega \ll \omega_0$. Note that the second time constant in the circuit RC is at an even higher frequency than the resonant frequency for $Q > 1$.

For more complex structures, such as inductor modeled on a lossy substrate such as silicon, one approach is to simply model each element with an equivalent circuit, as shown in Fig. 7.11a. Here we assume that the capacitor and resistor are nearly ideal but the inductor has parasitic capacitance and resistance, modeling the self-resonance and the conductor and substrate losses. Analyzing such a circuit is trivial but messy, leading to higher order equations containing many poles/zeros and it's unlikely to give us insight into the circuit behavior. An alternative approach to observe that no matter how complicated the inductor model, at a fixed frequency below the component resonance frequency, the net impedance of the inductor can be represented as $Z_L = R_{\text{eff}} + j\omega L_{\text{eff}}$. While this is obviously true, we will derive some simple formulas for quickly deriving this equivalent impedance from a complicated circuit model.

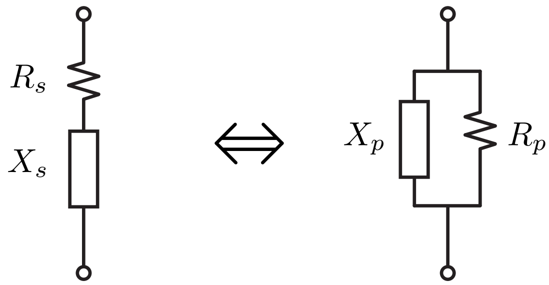


Figure 7.12: A series to parallel transformation for an arbitrary impedance Z.

Shunt-Series Transformation

The key calculation aid is the series to parallel transformation. Consider the impedance shown in Fig. 7.12, which we wish to represent as a parallel impedance. We can do this at a single frequency as long as the impedance of the series network equals the impedance of the shunt network

$$R_s + jX_s = \frac{1}{\frac{1}{R_p} + \frac{1}{jX_p}} \tag{7.75}$$

Equating the real and imaginary parts

$$R_s = \frac{R_p X_p^2}{R_p^2 + X_p^2} \tag{7.76}$$

$$X_s = \frac{R_p^2 X_p}{R_p^2 + X_p^2} \tag{7.77}$$

which can be simplified by using the definition of Q

$$Q_s = \frac{X_s}{R_s} = \frac{R_p^2 X_p}{R_p X_p^2} = \frac{R_p}{X_p} = Q_p = Q \tag{7.78}$$

which shows that

$$R_p = R_s(1 + Q^2) \tag{7.79}$$

and

$$X_p = X_s(1 + Q^{-2}) \approx X_s \tag{7.80}$$

where the approximation applies under high Q conditions.

Simplifying Practical *RLC* Resonators

The simplified RLC circuit appears in Fig. 7.11b, which has the same equivalent form as a second-order RLC circuit. Here the effective inductance L_{eff} of the circuit is different from L. If the circuit is operated far below the SRF of the inductor, then $L_{\text{eff}} \approx L$. The loss in the circuit now includes the effective series resistance and equivalent shunt substrate losses of the inductor. Again, at frequencies far below the SRF of the inductor, the resistance is dominated by the conductive loss of the inductor winding. We can thus conclude that for a practical RLC circuit employing non-ideal RLC elements, we can treat the circuit as a simple series RLC if all elements are used below their self-resonant frequency and represented as equivalent series impedances. The self-resonant frequency of the circuit is then determined by the net capacitance and inductance in the loop, and the Q factor is determined by the total series resistance in the loop. For this reason, series RLC circuits are sensitive to any series resistance in the circuit and the layout must be done properly in order to minimize any contact or interconnect resistance.

It is thus clear that the circuit Q is necessarily lower than the component Q_x, where Q_x is defined in the "series" sense. If the net impedance of a component is $Z = R + jX$, then

$$Q_x = \frac{X}{R} \tag{7.81}$$

At resonance the circuit Q is given by

$$Q = \frac{X'}{R + R_{x,L} + R_{x,C}} \tag{7.82}$$

where X' is the net capacitive reactance at resonance and $R_{x,L|C}$ represent the series losses in the reactive components. Given that typically $X' \approx X$, we can re-write the above as

$$\frac{1}{Q} \approx \frac{R + R_{x,1} + R_{x,2}}{X} = \frac{1}{Q_{id}} + \frac{1}{Q_L} + \frac{1}{Q_C} \tag{7.83}$$

where Q_{id} is the Q of the circuit using "ideal" components. This means that the circuit Q is approximately the parallel combination of the Q's

$$Q = Q_{id}||Q_L||Q_C < Q_{id} \tag{7.84}$$

Example 11:

Design a series RLC circuit with $R = 5\Omega$, using an inductor with $L = 5$ nH and component $Q_L = 30$ and a capacitor with component $Q_C = 200$ to resonate at 1 GHz.

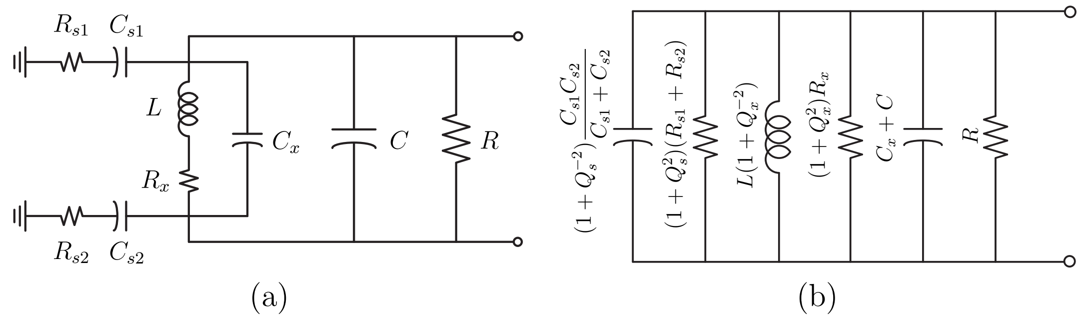


Figure 7.13: (a) A parallel RLC circuit containing a non-ideal inductor L. (b) A simplified equivalent circuit for the parallel RLC circuit.

We begin by calculating the equivalent series resistance of each component.

$$R_{x,L} = \frac{X_L}{Q_L} = \frac{31.416}{30} \approx 1.05\Omega$$

$$R_{x,C} = \frac{X_C}{Q_C} = \frac{31.416}{200} \approx 0.16\Omega$$

The total series resistance of the circuit is thus $R_T = 6.21\ \Omega$, resulting in a net $Q = 5$. In comparison, the unloaded $Q = 6.3$.

The exact same considerations apply for a parallel RLC circuit with the exception that now *shunt* parasitics de-Q the circuit. To see this, consider a non-ideal RLC circuit shown in Fig. 7.13. Now we convert each non-ideal element into an ideal admittance $Y = G_x + jB'$. The net conductance of the circuit is given by

$$G_T = G_{x,L} + G_{x,C} + G \tag{7.85}$$

The resulting equivalent circuit is shown in Fig. 7.13. For operation below the SRF of the components, therefore, Eq. 7.84 applies.

LC Tanks

In many applications, such as oscillators, we use an LC tank as a frequency reference. In an oscillator, for instance, the frequency of oscillation is determined by the frequency where the phase shift through the loop is a 0° (any multiple of 2π). As the phase selectivity of an RLC tank is determined by the Q, we maximize the "tank" Q by using the highest quality

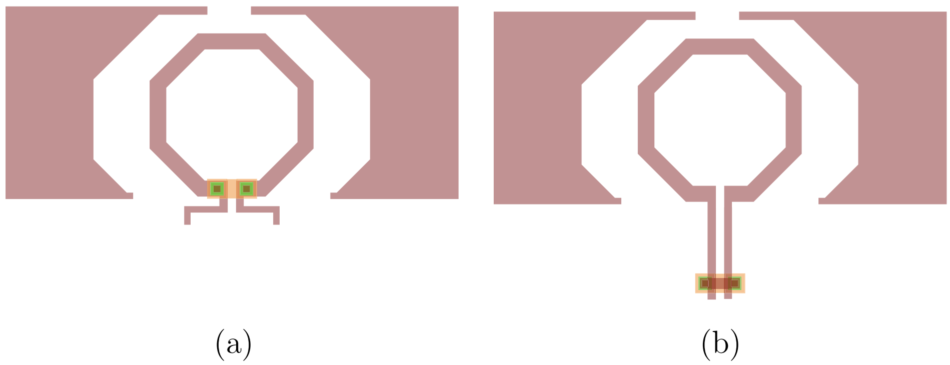


Figure 7.14: (a) An LC tank layout employing a ring inductor and two series connected MIM capacitors. (b) An inferior LC tank layout due to extra lead inductance and loss.

inductance and capacitance available. Furthermore, we eliminate or minimize the loading resistance so we omit R to obtain an LC tank. Any residual R in the parallel RLC tank is due to losses (component Q_c) or unwanted loading.

The tank impedance at resonance is therefore real and only limited by the realizable quality factor Q

$$Z(\omega_0) = R_p = \omega_0 L \times Q \tag{7.86}$$

The layout of an integrated LC tank circuit appears in Fig. 7.14a. In the layout of the LC tank in an IC environment, it's often easy to forget that every wire interconnect adds resistance and potentially couples to the substrate and thus it can de-Q the circuit. Here a small ring inductor resonates with MIM capacitors. An electric ground shield surrounds the ring without forming a closed loop. Notice that the MIM capacitors are placed directly underneath the leads of the inductor. The measured impedance of this structure is shown in Fig. 7.15. As expected, the circuit behaves as an LC circuit with the ring inductor losses dominating the Q. Despite the high frequency resonance of 56 GHz, the ring tank behaves extraordinarily like a simple lumped LC circuit. This follows because the dimensions of the structure are much smaller than wavelength over the operating range.

It's important to note that the layout of Fig. 7.14b is sub-optimal (but commonly used) as it adds unnecessary lead inductance and resistance to the parallel tank. Every effort should be made in the layout to put the capacitor leads as close to the inductor leads to minimize losses.

Notice that the layout in Fig. 7.14a uses two series MIM capacitors. This is done for two important reasons. Even though the net capacitance is the same as a single capacitor of half the size, larger capacitors have better matching and are more accurate. Furthermore, due to the differential excitation, the capacitor common node is not excited and acts as an AC ground. This is beneficial since the bottom plate is closer to the substrate and thus is

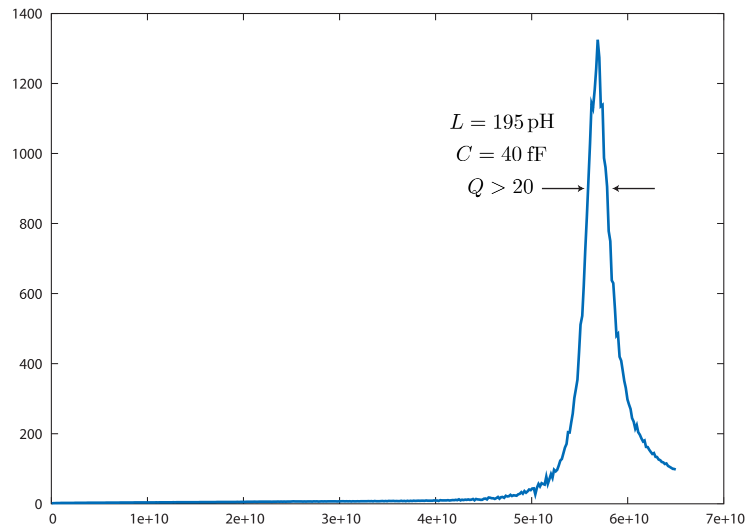


Figure 7.15: The on-chip measured LC tank impedance.

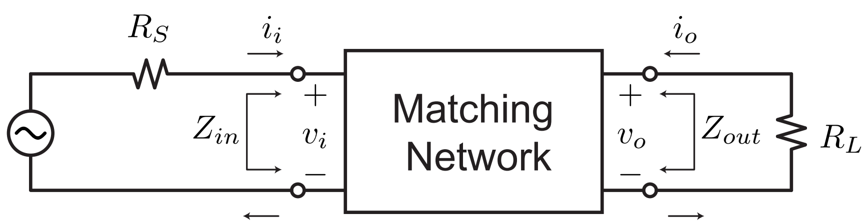


Figure 7.16: A generic matching network as a black box.

a lossier node.

7.3 Impedance Matching

In the words of one experienced designer, "RF design is all about impedance matching." In this section we'd like to show how inductors and capacitors are handy elements at impedance matching. Viewed as a black-box shown in Fig. 7.16, an impedance matcher changes a given load resistance R_L to a source resistance R_S. Without loss of generality, assume $R_S > R_L$, and a power match factor of $m = R_S/R_L$ is desired. In fact any matching network that boosts the resistance by some factor can be flipped over to do the opposite matching.

Since $R_L = v_o/i_o$ and $R_S = v_i/i_i$, we can see that this transformation can be achieved by a voltage gain, $v_i = kv_o$. Assuming the black box is realized with passive elements without memory, power conservation implies

$$i_i v_i = i_o v_o \tag{7.87}$$

thus the current must drop by the same factor, $i_i = k^{-1} i_o$, resulting in

$$Z_{in} = \frac{v_i}{i_i} = \frac{k v_o}{k^{-1} i_o} = k^2 \frac{v_o}{i_o} = k^2 R_L \tag{7.88}$$

which means that $k = \sqrt{m}$ to achieve an impedance match. There are many ways to realize such a circuit block and in this section we'll explore techniques employing inductors and capacitors. In Ch. 10 we'll explore transformer based impedance matching.

Why Play the Matchmaker?

Optimal Power Transfer

Perhaps the most important reason for matching is to maximize the power transfer from a source to a load. Recall that the maximum power available form a fixed voltage source with impedance Z_S is obtained when the load is the complex conjugate impedance, or $Z_L = Z_S^*$. To see that such an optimum impedance must exists notice that the power delivered is zero when an open or short is attached to the source and non-zero for any practical value of the load. By continuity we can see that the curve of P_L has a maximum.

Optimal Noise Figure

There are other reasons to match impedances and we'll mention them in passing. Another important case is to minimize the noise figure of an amplifier driven for a given source impedance. It can be shown that the noise figure of a two-port (e.g. a single transistor or a multi-stage amplifier) takes on the following general form

$$F = F_{min} + \frac{R_n}{G_g} |Y_s - Y_{s,opt}|^2 \tag{7.89}$$

The parameters R_n, G_g, and $Y_{s,opt}$ are properties of the amplifier at a particular bias point. The noise figure is thus minimized when the source impedance equal to $Y_{s,opt}$. So another function for a matching network is to perform this transformation. In fact, in a modern integrated circuit one has control over the physical dimension of the transistor and thus one can size a transistor appropriately so that a noise match corresponds to an optimal power gain match.

Minimum Reflections in Transmission Lines

Another important reason for impedance matching will become evident when we study switching transients on a transmission line (Ch. 12). Here impedance matching is more commonly known as "terminating" a transmission line. Without a proper termination, reflections can lead to inter-symbol interference. A practical example is a ghost of a television screen caused by reflections off the antenna and television amplifier. These reflections travel back and forth on the feed-line and produce weak secondary copies of the signals, which appear as ghosts on the screen.

Optimal Efficiency

Power amplifiers present more reasons to match impedances. In particular, while an impedance match results in maximum power transfer, an entirely different impedance is needed to achieve *optimal efficiency*. To see this observe that the drain efficiency of an amplifier can be written as

$$\eta = \frac{P_L}{P_{dc}} = \frac{\frac{1}{2}v_o i_o}{I_Q V_{sup}} \tag{7.90}$$

where I_Q is the average current drawn from the supply voltage V_{sup} over one cycle and v_o and i_o are the output voltage and current swings in the amplifier. Re-writing the above equation into normalized form

$$\eta = \frac{1}{2}\hat{I}\hat{V} \tag{7.91}$$

where $\hat{I} = i_o/I_Q$ is the normalized current swing and $\hat{V} = v_o/V_{sup}$ is the normalized voltage swing. By conservation of energy $\hat{I} \times \hat{V} \leq 2$. For a tuned amplifier, $\hat{V} \leq 1$ and $\hat{I} \leq 2$. Once these values have been specified to achieve a certain efficiency, the value of the load impedance is fixed by the ratio

$$R_{L,opt} = \frac{v_o}{i_o} = \frac{\hat{V}}{\hat{I}} \times \frac{V_{sup}}{I_Q} \tag{7.92}$$

Let's do an example. Say we'd like to deliver 1 W of power in a modern CMOS process with $V_{sup} < 2$ V (due to breakdown). Since the maximum voltage swing cannot exceed the supply ($\hat{V} \leq 1$), the required current swing is found from the required output power

$$\tfrac{1}{2} i_o v_o = P_L \tag{7.93}$$

or $i_o \geq 2P_L/V_{sup}$, which works out to $i_o \geq 1$ A. The optimal load impedance is thus as low as $2\text{V}/1\text{A} = 2\ \Omega$, a very small resistance. The load in most communication systems is the the system impedance Z_0, usually $Z_0 = 50\Omega$ (RF) or 75Ω (video), or even a higher impedance. Clearly an impedance matching network is needed to to convert the load impedance Z_0 to 2Ω to achieve the required output power and optimal efficiency.

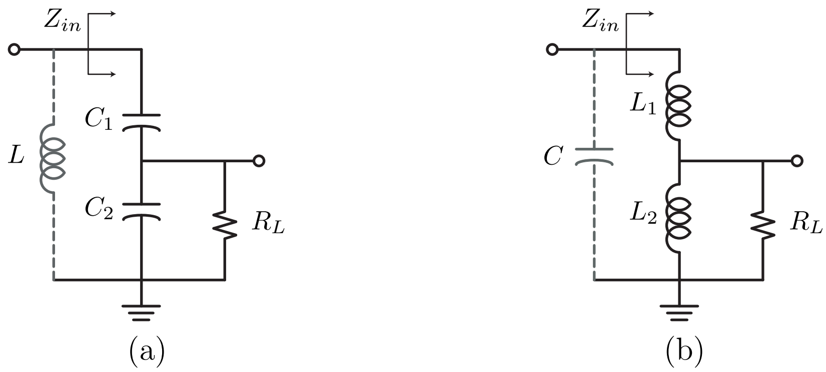


Figure 7.17: (a) A tapped capacitive divider impedance transformer. (b) A tapped inductor impedance transformer. The reactance in the structure can be resonated with an appropriate elements (shown in dashed line).

Capacitive and Inductive Dividers

Perhaps the simplest matching networks are simple voltage dividers. Consider the capacitive voltage divider shown in Fig. 7.17a. At RF frequencies, if $R_L \gg X_2$, then we can see that the circuit will work as advertised. Assuming that negligible current flows into R_L, the current flowing into the capacitors is given by

$$i = \frac{v_i}{j(X_1 + X_2)} \tag{7.94}$$

the voltage across the is therefore

$$v_o = v_{C_2} = jX_2 \times i = v_i \frac{X_2}{X_1 + X_2} = v_i \frac{1}{1 + \frac{C_2}{C_1}} = kv_i \tag{7.95}$$

which means that the load resistance is boosted by a factor of k^2

$$R_{in} \approx \left(1 + \frac{C_2}{C_1}\right)^2 R_L \tag{7.96}$$

We can arrive at the same destination by using the shunt $\leftrightarrow$ series transformation twice. The final value of R_{in} is given by a $1 + Q_2^2$ reduction following by a $1 + Q_s^2$ enhancement

$$R_{in} = \frac{1 + Q_s^2}{1 + Q_2^2} R_L \tag{7.97}$$

where $Q_2 = \frac{R_L}{X_2}$, $X_s = X_1 || X_2'$, and

$$Q_s = \frac{X_s}{R_L}(1 + Q_2^2) \tag{7.98}$$

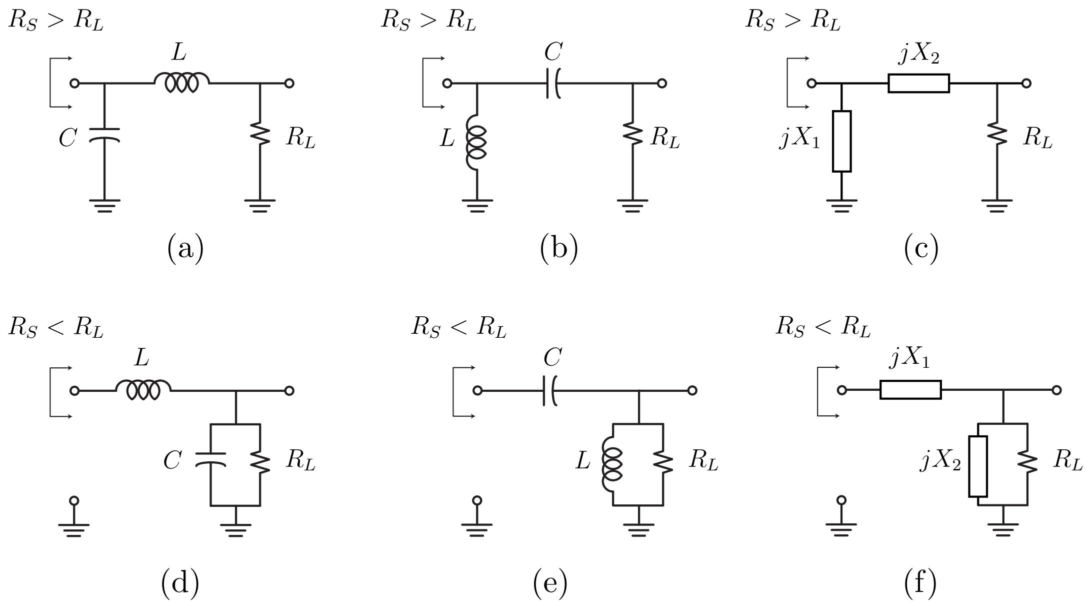


Figure 7.18: Several incarnations of L-matching networks. In (a)-(c) the load is connected in series with the reactance boosting the input resistance. In (d)-(f) the load is in shunt with the reactance, lowering the input resistance.

The final expression is derived after some algebra

$$R_{in} = \frac{R_L}{1+Q_2^2} + \left(\frac{X_s}{R_L}\right)^2 + \left(\frac{X_s}{X_2}\right)^2 R_L \tag{7.99}$$

Under the assumption that $X_2 \ll R_L$, the final term dominates

$$R_{in} = \left(\frac{X_s}{X_2}\right)^2 R_L \approx \left(1 + \frac{C_2}{C_1}\right)^2 R_L \tag{7.100}$$

as expected. The reactance of the capacitive divider can be absorbed by a resonating inductance as shown in Fig. 7.17. In a similar vein, an inductive divider matching circuit can be designed as shown in Fig. 7.17.

An *L*-Match

Consider the L-Matching networks, shown in Fig. 7.18, named due to the topology of the network. We shall see that one direction of the L-match boosts the load impedance (in series with load) whereas the other lowers the load impedance (in shunt with the load). Let's focus on the first two networks shown in Fig. 7.18ab. Here, in absence of the source, we have a

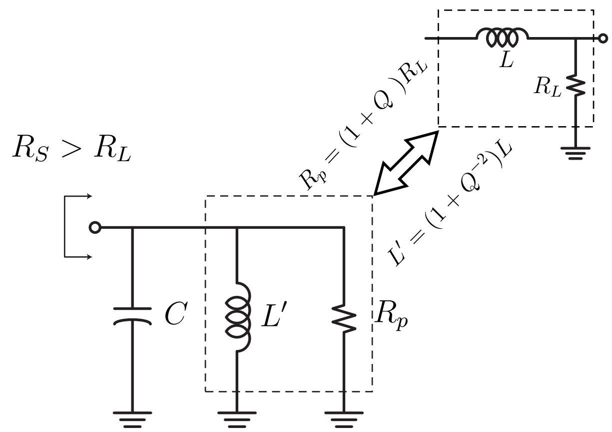


Figure 7.19: The transformed L matching network into a parallel RLC equivalent circuit.

simple series RLC circuit. Recall that in resonance, the voltage across the reactive elements is Q times larger than the voltage on the load! In essence, that is enough to perform the impedance transformation. Without doing any calculations, you can immediately guess that the impedance seen by the source is about Q^2 larger than R_L. Furthermore, since the circuit is operating in resonance, the net impedance seen by the source is purely real. To be sure, let's do the math.

A quick way to accomplish this feat is to begin with the series to parallel transformation, as shown in Fig. 7.19, where the load resistance in series with the inductor is converted to an equivalent parallel load equal to

$$R_p = (1 + Q^2)R_L \tag{7.101}$$

where $Q = X_L/R_L$, and $X'_L = X_L(1+Q^{-2})$. The circuit is now nothing but a parallel RLC circuit and it's clear that at resonance the source will see only R_p, or a boosted value of R_L. The boosting factor is indeed equal to $Q^2 + 1$, very close to the value we guessed from the outset.

To gain insight into the operation of Fig. 7.18d-f, consider an Norton equivalent of the same circuit shown in Fig. 7.20. Now the circuit is easy to understand since it's simply a parallel resonant circuit. We known that at resonance the current through the reactances is Q times larger than the current in the load. Since the current in the series element (L in Fig. 7.18d) is controlled by the source voltage, we can immediately see that $i_s = Qi_L$, thus providing the required current gain to lower the load resistance by a factor of Q^2.

As you may guess, the mathematics will yield a similar result. Simply do a parallel to series transformation of the load to obtain

$$R_s = \frac{R_p}{1 + Q^2} \tag{7.102}$$

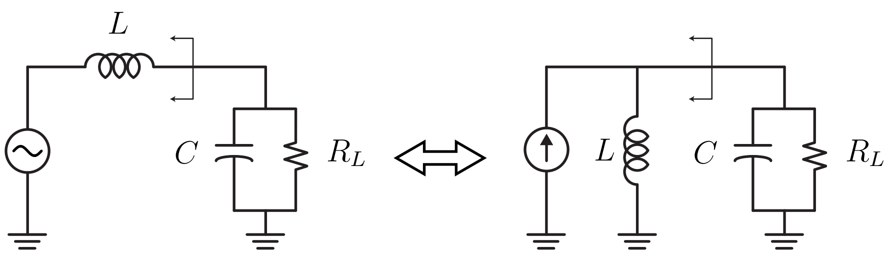


Figure 7.20: The source voltage driving the L matching network can be transformed into an equivalent Norton current source.

$$X_p' = \frac{X_p}{1 + Q^{-2}} \tag{7.103}$$

The resulting circuit is a simple series RLC circuit. At resonance, the source will only see the reduced series resistance R_s.

L-Match Design Equations

The following design procedure applies to an L-match using the generic forms of Fig. 7.18c,f. The actual choice between Fig. 7.18a,d and Fig. 7.18b,e depends on the application. For instance Fig. 7.18b,e provide AC coupling (DC isolation) which may be required in many applications. In other applications a common DC voltage may be needed, making the networks of Fig. 7.18a,d the obvious choice.

Let $R_{hi} = \max(R_S, R_L)$ and $R_{lo} = \min(R_S, R_L)$. The L-matching networks shown in Fig. 7.18 are designed as follows:

1. Calculate the boosting factor $m = \frac{R_{hi}}{R_{lo}}$.
2. Compute the required circuit Q by $(1 + Q^2) = m$, or $Q = \sqrt{m - 1}$.
3. Pick the required reactance from the Q. If you're boosting the resistance, e.g. $R_S > R_L$, then $X_s = Q \cdot R_L$. If you're dropping the resistance, $X_p = \frac{R_L}{Q}$.
4. Compute the effective resonating reactance. If $R_S > R_L$, calculate $X_s' = X_s(1 + Q^{-2})$ and set the shunt reactance in order to resonate, $X_p = -X_s'$. If $R_S < R_L$, then calculate $X_p' = \frac{X_p}{1+Q^{-2}}$ and set the series reactance in order to resonate, $X_s = -X_p'$.
5. For a given frequency of operation, pick the value of L and C to satisfy these equations.

Insertion Loss of an *L*-Matching Network

We'd like to include the losses in our passive elements into the design of the matching network. The most detrimental effect of the component Q is the insertion loss which reduces the power transfer from source to load.

Let's begin by using our intuition to derive an approximate expression for the loss. Note that the power delivered to the input of the matching network P_{in} can be divided into two components

$$P_{in} = P_L + P_{diss} \tag{7.104}$$

where P_L is the power delivered to the load and P_{diss} is the power dissipated by the non-ideal inductors and capacitors. The insertion loss is therefore given by

$$IL = \frac{P_L}{P_{in}} = \frac{P_L}{P_L + P_{diss}} = \frac{1}{1 + \frac{P_{diss}}{P_L}} \tag{7.105}$$

Recall that for the equivalent series RLC circuit in resonance, the voltages across the reactances are Q times larger than the voltage across R_L. We can show that the reactive power is also a factor of Q larger. For instance the energy in the inductor is given by

$$W_m = \frac{1}{4} L i_s^2 = \frac{1}{4} \frac{v_s^2}{4R_S^2} L \tag{7.106}$$

or

$$\omega_0 \times W_m = \tfrac{1}{4} \frac{v_s^2}{4R_S} \frac{\omega_0 L}{R_S} = \tfrac{1}{2} \frac{v_s^2}{8R_S} Q = \tfrac{1}{2} P_L \times Q \tag{7.107}$$

where P_L is the power to the load at resonance

$$P_L = \frac{v_L^2}{2R_S} = \frac{v_s^2}{4 \cdot 2 \cdot R_S} = \frac{v_s^2}{8R_S} \tag{7.108}$$

The total reactive power is thus exactly Q times larger than the power in the load

$$\omega_0 (W_m + W_e) = Q \times P_L \tag{7.109}$$

By the definition of the component Q_c factor, the power dissipated in the non-ideal elements of net quality factor Q_c is simply

$$P_{diss} = \frac{P_L \cdot Q}{Q_c} \tag{7.110}$$

which by using Eq. 7.105 immediately leads to the following expression for the insertion loss

$$IL = \frac{1}{1 + \frac{Q}{Q_c}} \tag{7.111}$$

The above equation is very simple and insightful. Note that using a higher network Q, e.g. a higher matching ratio, incurs more insertion loss with the simple single stage matching network. Furthermore, the absolute component Q is not important but only the component Q_c normalized to the network Q. Thus if a low matching ratio is needed, the actual components can be moderately lossy without incurring too much insertion loss.

Also note that the the actual inductors and capacitors in the circuit can be modeled with very complicated sub-circuits, with several parasitics to model distributed and skin effect, but in the end, at a given frequency, one can calculate the equivalent component Q_c factor and use it in the above equation.

Note that Q_c is the net quality factor of the passive elements. If one element dominates, such as a low-Q inductor, then Q_L can be used in its place. The exact analysis for a lossy inductor and capacitor is simple enough and yields an expression that is identical to Eq. 7.111 when only inductor losses are taken into account but differs when both inductor and capacitors losses are included

$$IL = \frac{1}{1+\frac{Q}{Q_L}}\frac{1}{1+\frac{Q}{Q_C}} \tag{7.112}$$

which can be written as

$$IL = \frac{1}{1+Q\left(Q_L^{-1}+Q_C^{-1}\right)+\frac{Q^2}{Q_L Q_C}} \tag{7.113}$$

which equals the general expression we derived under "high-Q" conditions, e.g. $Q_c \gg Q$.

When completing the design with real elements, it's also necessary to shift the component values slightly due to the extra loss. While formulas for these perturbations can be calculated, a modern computer and optimizer really make this exercise unnecessary.

Reactance Absorption

In most situations the load and source impedances are often complex and our discussion so far only applies to real load and source impedances. An easy way to handle complex loads is to simply absorb them with reactive elements. For example, consider the complex load shown in Fig. 7.21. To apply an L-matching circuit, we can begin by simply resonating out the load reactance at the desired operating frequency. For instance, we add an inductance L_{res} in shunt with the capacitor to produce a real load. From here the design procedure is identical. Note that we can absorb the inductor L_{res} into the shunt L-matching element.

From now onwards we can simply discuss the real matching problem since a complex load or source can be handled in a similar fashion. Often there are multiple ways to perform the absorption with each choice yielding slightly different network properties such as Q (bandwidth), and different frequency selectivity (e.g. low-pass, high-pass, bandpass).

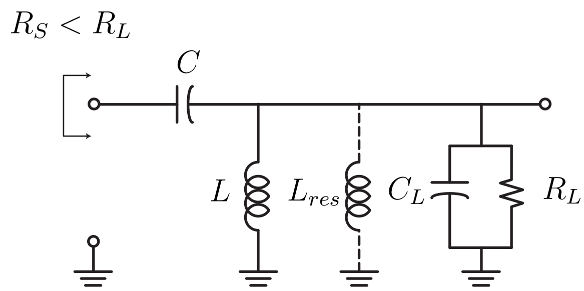


Figure 7.21: The complex load C_L in parallel with R_L is matched to a real source impedance by first applying an parallel inductor L_{res} to resonate out C_L. The load can now be matched using a standard matching network. In the final design, the resonating L_{res} can be simply absorbed into L.

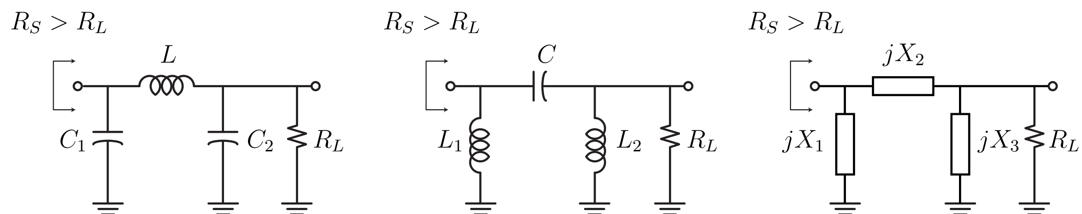


Figure 7.22: Several incarnations of a Π matching network. The first is a low-pass structure, the second a high-pass structure. The third is a general Π network.

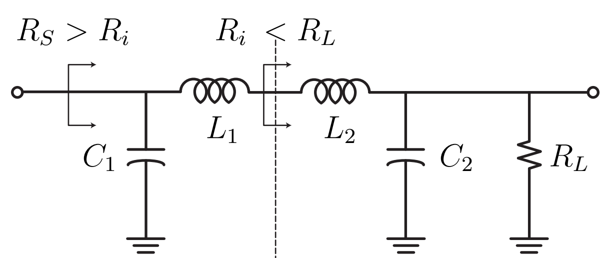


Figure 7.23: The Π network can be decomposed into a back-to-front cascade of two L matching networks. The impedance is first reduce down to $R_i < R_L$, then increased back up to $R_S > R_L > R_i$.

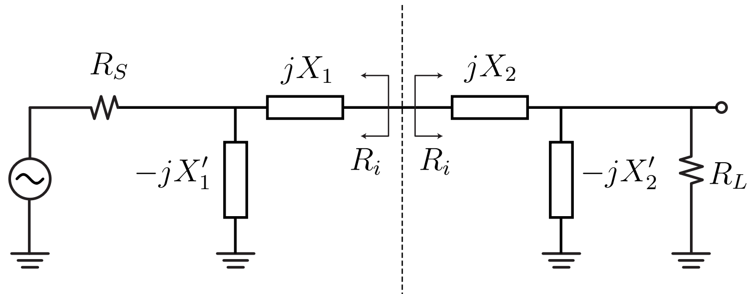


Figure 7.24: The reflected input and output impedance are both equal to R_i at the center of the Π network.

A Π-Match

The L-Match circuit is simple and elegant but is somewhat constrained. In particular, we cannot freely choose the Q of the circuit since it is fixed by the required matching factor m. This restriction is easily solved with the Π-Matching circuit, also named from its topology, shown in Fig. 7.22. The idea behind the Π match can be easily understood by studying the cascade of two back-to-front L matches as shown in Fig. 7.23. In this circuit the first L match will lower the load impedance to an intermediate value R_i

$$R_i = \frac{R_L}{1 + Q_1^2} \tag{7.114}$$

or

$$Q_1 = \sqrt{\frac{R_L}{R_i} - 1} \tag{7.115}$$

Since $R_i < R_L$, the second L match needs to boost the value of R_i up to R_s. The Q of the second L network is thus

$$Q_2 = \sqrt{\frac{R_S}{R_i} - 1} > \sqrt{\frac{R_S}{R_L} - 1} \tag{7.116}$$

When we combine the two L networks, we obtain a Π network with a higher Q than possible with a single stage transformation. In general the Q, or equivalently the bandwidth $B = \frac{\omega_0}{Q}$, is a free parameter that can be chosen at will for a given application. Note that when the source is connected to the input, the circuit is symmetric about the center, as shown in Fig. 7.24. Now it's rather easy to compute the network Q by drawing a series equivalent circuit about the center of the structure, as shown in Fig. 7.25. If the capacitors and inductors in series are combined, the result is a simple RLC circuit with Q given by

$$Q = \frac{X_1 + X_2}{2R_i} = \frac{Q_1 + Q_2}{2} \tag{7.117}$$

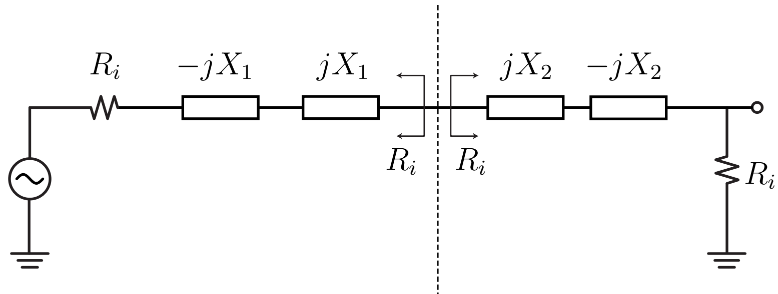


Figure 7.25: The L sections can be converted into series sections to produce one big LCR circuit.

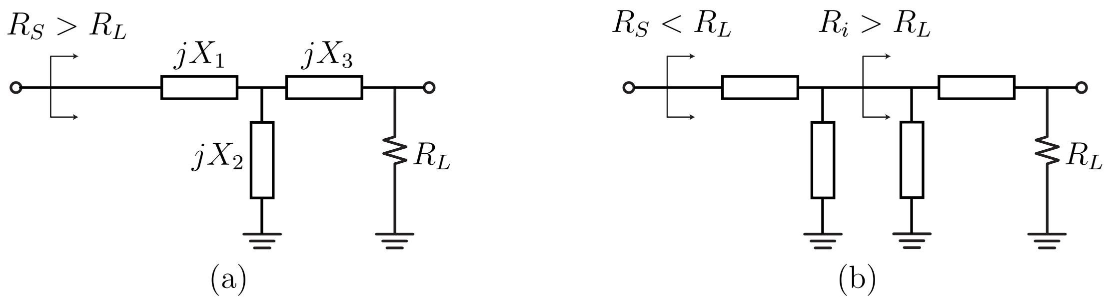


Figure 7.26: (a) The T-matching network can be decomposed into two front-to-back L sections. (b) The first L section boosts the resistance to a value of $R_i > R_L$ and the second L structure drops the impedance to $R_S < R_i$.

It's important to note the inclusion of the source resistance when calculating the network Q as we are implicitly assuming a power match. In a power amplifier, the source impedance may be different and the above calculation should take that into consideration. For instance, if the PA is modeled as a high impedance current source (Class A/B operation), then the factor of 2 disappears. The design procedure begins with the specification of the network Q. Eq. 7.117 is then used to find R_i, and from there the L-match procedure outlined above takes over.

A *T*-Match

The T-matching network, shown in Fig. 7.26a, is the dual of the Π network. By now you can see that the names all correspond the physical topology of the circuit. The T network can also be decomposed into a cascade of two back-to-front L networks, as shown in Fig. 7.26b. The first L transforms the resistance up to some intermediate value $R_i > R_S$, and the second L transforms the resistance back down to R_S. Thus the net Q is higher than a single stage

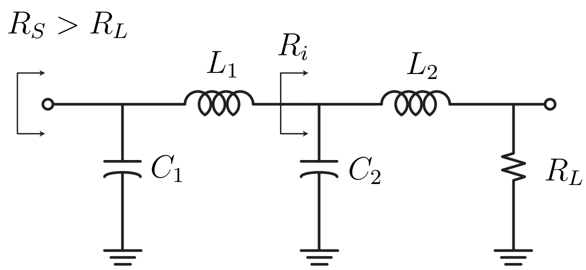


Figure 7.27: A two-stage low-pass L matching network. The first stage steps up the intermediate resistance $R_S < R_i < R_L$, thus lowering the Q over a single stage design.

match. The network Q can be derived in an analogous fashion and yields the same solution

$$Q = \tfrac{1}{2}\left(\sqrt{\frac{R_i}{R_L} - 1} + \sqrt{\frac{R_i}{R_S} - 1}\right) \tag{7.118}$$

Multi-Section Low Q Matching

We have seen that the Π and T matching networks are essentially two stage networks which can boost the network Q. In many applications we actually would like to achieve the opposite effect, e.g. low network Q is desirable in broadband applications. Furthermore, a low Q design is less susceptible to process variations. Also, a lower Q network lowers the loss of the network, as evident by examining Eq. 7.111.

To lower the Q of an L matching network, we can employ more than one stage to change the impedance in smaller steps. Recall that $Q = \sqrt{m-1}$, and a large m factor requires a high Q match. If we simply change the impedance by a factor $k < m$, the Q of the first L section is reduced. Likewise, a second L section will further change the resistance to the desired R_S with a step size $l < m$, where $l \cdot k = m$. An example two-stage network is shown in Fig. 7.27a. Reflecting all impedances to the center of the network, the real part of the impedance looking left or right is R_i at resonance. Thus the power dissipation is equal for both networks. The overall Q is thus given by

$$Q = \frac{\omega(W_{s1} + W_{s2})}{P_{d1} + P_{d2}} = \frac{\omega W_{s1}}{2P_d} + \frac{\omega W_{s2}}{2P_d} = \frac{Q_1 + Q_2}{2} \tag{7.119}$$

$$Q = \tfrac{1}{2}\left(\sqrt{\frac{R_i}{R_L} - 1} + \sqrt{\frac{R_S}{R_i} - 1}\right) \tag{7.120}$$

Note the difference between the above and Eq. 7.118. The R_i term appears once in the denominator and once in the numerator since it's an intermediate value. What's the lowest

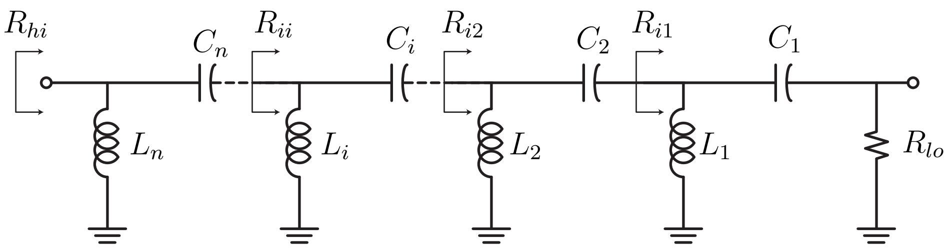


Figure 7.28: A high-pass multi-section L matching network.

Q achievable? To find out, take the derivative of 7.120 with respect to R_i and solve for the minimum

$$R_{i,\text{opt}} = \sqrt{R_L R_S} \tag{7.121}$$

which results in a Q approximately lower by a square root factor

$$Q_{opt} = \sqrt{\sqrt{\frac{R_S}{R_L}} - 1} \approx m^{1/4} \tag{7.122}$$

It's clear that the above equations apply to the opposite case when $R_L > R_S$ by simply interchanging the role of the source and the load.

To even achieve a lower Q, we can keep adding sections as shown in Fig. 7.28. The optimally low Q value is obtained when the intermediate impedances are stepped in geometric progression

$$\frac{R_{i1}}{R_{lo}} = \frac{R_{i2}}{R_{i1}} = \frac{R_{i3}}{R_{i2}} = \cdots = \frac{R_{hi}}{R_{in}} = 1 + Q^2 \tag{7.123}$$

where $R_{hi} = \max(R_S, R_L)$ and $R_{lo} = \min(R_S, R_L)$. In the limit that $n \to \infty$, we take very small "baby" steps from R_{lo} to R_{hi} and the circuit starts to look like a tapered transmission line. Multiplying each term in the above equation

$$\frac{R_{i1}}{R_{lo}} \cdot \frac{R_{i2}}{R_{i1}} \cdot \frac{R_{i3}}{R_{i2}} \cdot \cdots \cdot \frac{R_{hi}}{R_{in}} = \frac{R_{hi}}{R_{lo}} = (1 + Q^2)^N \tag{7.124}$$

which results in the optimally Q factor for the overall network

$$Q = \sqrt{\left(\frac{R_{hi}}{R_{lo}}\right)^{1/N} - 1} \tag{7.125}$$

The loss in the optimal multi-section line can be calculated as follows. Using the same approach as Sec. 7.3, note that the total power dissipated in the matching network is given by

$$P_{diss} = \frac{NQP_L}{Q_u} \tag{7.126}$$

where N section are used, each with equal Q due to the condition set forth by Eq. 7.123. This leads to the following expression

$$IL = \frac{1}{1 + N\frac{Q}{Q_u}} \tag{7.127}$$

or

$$IL = \frac{1}{1 + \frac{N}{Q_u}\sqrt{\left(\frac{R_{hi}}{R_{lo}}\right)^{1/N} - 1}} \tag{7.128}$$

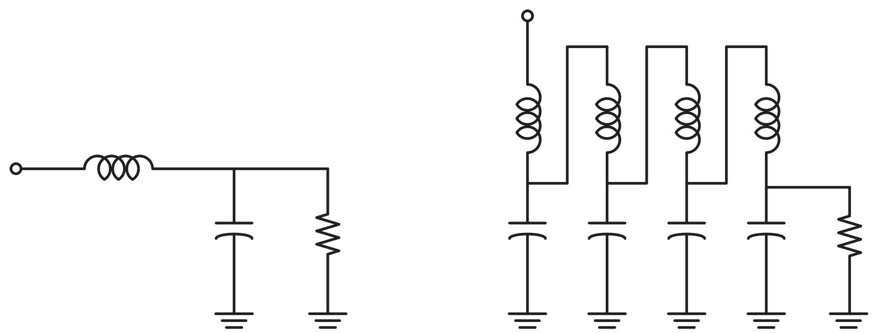
It's interesting to observe that this expression has an optimum for a particular value of N. It's easy enough to plot IL for a few values of N to determine the optimal number of sections. Intuitively adding sections can decrease the insertion loss since it also lowers the network Q factor. Adding too many sections, though, can counterbalance this benefit.

Example 12:

Suppose a power amplifier delivering 100 W of power has an optimal load resistance of .5Ω, but needs to drive a 50Ω antenna. Design a matching network assuming that the component Q's of 30 are available.

First note that a matching factor of $m = 50/.5 = 100$ is needed. The table below shows the network Q and insertion loss as a function of the number of sections N. Clearly three sections yields the optimal solution. But since a three section filter is more expensive, and has only marginally better performance, a two section matching network may be preferable.

N	Q (Eq. 7.125)	IL (dB) (Eq. 7.128)
1	9.95	−1.24
2	3	−0.79
3	1.91	−0.76
4	1.47	−0.78
5	1.23	−0.81
6	1.07	−0.85



(a) (b)

Figure 7.29: (a) A single stage LC voltage divider has a low-pass transfer characteristic. Its topology identical to a L matching network. (b) Cascades of voltage dividers are equivalent to cascades of L sections leading to higher out of band attenuation.

7.4 Distributed Matching Networks

Matching circuits employing transmission lines are covered in Ch. 9. Quarter wave transmission lines, transmission line stubs, and other distributed structures will be covered. Transformer based matching, including transmission line transformers, are covered in Ch. 10 and Ch. 11.

7.5 Filters

A filter is a circuit with a specified frequency response characteristic. Filters are key elements in high speed communication circuits since often our information is buried in a lot of noise and interference. A filter can attenuate out-of-band signals and thus reduce the required dynamic range of analog and digital circuits, which leads directly to power savings (e.g. lower resolution ADC and fewer bits in the DSP).

Matching networks and filters have much in common. We may view a matching network as a filter with different load and source impedances. In effect all the circuits we have met in this chapter are filters, with the important distinction that while some filtering action was occurring, it was not a well controlled part of the design.

LC filters grow naturally from a simple low-pass LC dividers shown in Fig. 7.29a. At low frequencies, the inductor is a short and the capacitor is an open and so the input passes to the output unattenuated. At higher frequencies, though, the inductor tends to increase in reactance and the capacitance shunts the output to ground. How do we increase the out-of-band attenuation? Why not simply cascade another LC filter section as shown in Fig. 7.29b. In fact, we can continue to add sections to increase the out-of-band attenuation.

A high-pass version of this circuit is the dual where the capacitors and inductors switch

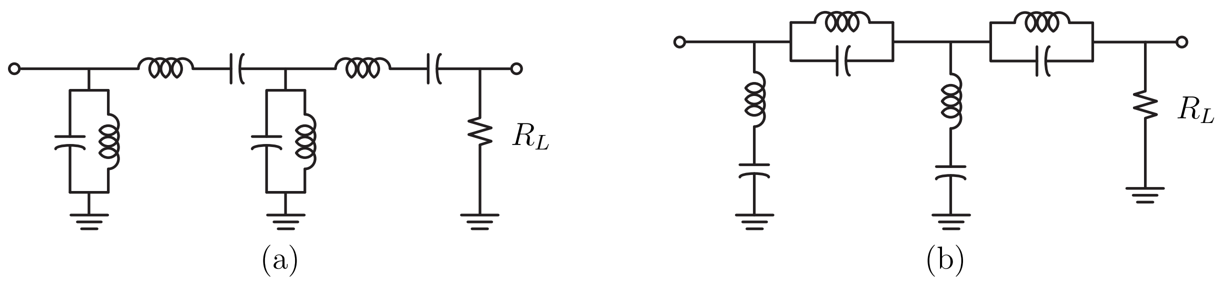


Figure 7.30: (a) A bandpass multi-section filter. (b) A multi-section notch filter.

places, as shown in Fig. 7.28. A bandpass version of the circuit is also easy if we observe that a series LC circuit is a short at resonance and a shunt LC circuit is an open at resonance. Then the circuit of Fig. 7.30a acts like a bandpass filter. If we switch the locations of the series and parallel resonant tanks, we obtain the network shown in Fig. 7.30b, a notch filter. Now the circuit response is a null at resonance.

The design of prototype filters is a very well developed discipline and many good sources exists [69] that tabulate filters into families with distinct characteristics (say minimum ripple or linear phase response).

7.6 References

This chapter has origins in lecture notes I used in several communication circuits courses taught at Berkeley. I learned this material from classic references, such as *Communication Circuits: Analysis and Design* [8] and *Solid State Radio* [31]. A more recent addition to the bookshelf, Tom Lee's, *The Design of CMOS RFIC* [34] book also has an excellent chapter covering some of this material.